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A LIMITED STUDY OF THE APPLICATION OF CONCENTRATED LOADS TO FIXED EDGE AND DISCONTINUOUS TWO-WAY SLABS OF REINFORCED CONCRETE, USING THE CENTRAL DIFFERENCE OPERATOR OF THE FINITE DIFFERENCE APPROXIMATION TO THE BIHARMONIC EQUATION

By

Ronnie Ray Henk

Report submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in partial fulfillment of the requirements for the degree of

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in

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Blacksburg, Virginia

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ABSTRACT

This paper makes a comparison of resulting design moments for a particular size, uniformly loaded, two-way reinforced concrete slab with various boundary conditions, using both the Central Difference Operator of the Finite Difference Approximation to the Biharmonic Equation and Method 3 of ACI Bulletin 318-63. The attempt of this study was to determine whether the large grid Finite Difference Analysis would reasonably approximate the results of ACI 318-63 Method 3 for a uniformly loaded slab.

Due to the reasonably close correlation of the results of the methods the Finite Difference Analysis, with the aid of the Elastic Curve Plot, was used to determine the design moments of the slab under the influence of a concentrated load. A concentrated load in Finite Difference Analysis is defined as being uniformly distributed over an equivalent grid area, with the center of the loaded area and the loaded grid point being coincident.

A method of estimating the maximum allowable, symmetrically located concentrated load is presented, although the results are unproven by testing.

A step-by-step account of the solution to each problem is presented.

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III. LIST OF FIGURES AND TABLES

IV. LIST OF SYMBOLS

A	area of the slab over which the live load is appli
A_{total}	total slab area
C	an assigned constant for the problem equal to $\frac{qh^4}{D}$
$C_{subscript}$	moment coefficient for two-way slabs as given in tables 1, 2, and 3 of ACI 318-63, Section A2003. Coefficients have identifying indexes, such as $C_{A\ neg}$, $C_{A\ LL}$, $C_{B\ DL}$, . . .
D	flexural rigidity of the plate equal to $\frac{Et^3}{12(1 - \mu^2)}$
DL	dead load
h	grid spacing
LL	live load
m	ratio of short span to long span for two-way slabs
M	moment, in ft-lb. Identifying indexes refer to direction, sign, and type of loading, such as $M_{x\ pos\ LL}$, $M_{y\ neg\ DL}$, . . .
P	an assigned constant for the problem equal to $\frac{q_t h^4}{D}$
pcf	lb per cubic ft
psf	lb per square ft
q	uniform load for entire slab, has identifying indexes such as q_{DL} and q_{LL}
q_a	concentrated total live load distributed over one grid area, equal to $\frac{W}{h^2}$

q_t	concentrated total live load distributed over nine grid areas, equal to $\frac{W}{(3h)^2}$
R	an assigned constant for the problem equal to $\frac{q_a h^4}{D}$
t	slab thickness
T	allowable concentrated load
w	deflection
W	total live load
x,y	rectangular coordinate axes
X	length of clear span in short direction
Y	length of clear span in long direction
μ	Poisson's ratio

V. INTRODUCTION

Justification for this Study: ACI Bulletin 318-63[1] has been extablished as the code which "provides minimum requirements for the design and construction of reinforced concrete or composite structural elements of any structure erected under the requirements of the general building code of which this code forms a part." It presents the requirements for the design of a two-way reinforced concrete slab with varying boundary conditions, but only for the application of a uniform load. In practice, a uniform loading condition is seldom the usage condition. The ACI Bulletin 318-63 and authors of books of reinforced concrete design occasionally mention that other techniques can be used to analyze a two-way slab for non-uniform loading conditions. They also indicate that even though these methods give approximate answers, the answers are often within the realm of reasonably predicting the action of a slab structure.

A design engineer is often faced with the problem of determining the capability of a two-way slab to withstand a large concentrated load. Since ACI 318-63 does not handle this loading condition, and since ACI 318-63 is the basis for the reinforced concrete design, one must be certain that any other analytical method used will produce reasonably comparable answers. The method used should also be fairly straightforward so that the work may be done efficiently.

Intent of this Study: To a limited degree, this study will investigate the use of the Central Difference Operator of the Finite

Difference Approximation to the Biharmonic Equation to ascertain whether the resulting moments are comparable to ACI 318-63 Method 3. A large grid, suitable for desk calculator solution, will be used to establish a suitable efficiency in the analysis.

This study will begin by considering a particular two-way reinforced concrete slab of dimensions 20 ft by 20 ft, and determining the maximum moments for each of three cases of boundary conditions: Case "A," all edges fixed; Case "B," two opposite edges fixed, two opposite edges pinned; Case "C," all edges pinned. The initial determination will be made utilizing a uniform loading condition and analyzing by both ACI 318-63 Method 3 and by the Finite Difference Approximation to the Biharmonic Equation. A comparison of the resulting maximum moments will then be made. If a reasonably close correlation exists between the results of the two methods, then a reasonable degree of accuracy can be expected from the Finite Difference Approximation to the Biharmonic Equation to analyze the slab for the application of concentrated loads. In the Finite Difference Analysis, a concentrated load is defined as being uniformly distributed over an equivalent grid area ($h \times h$) with the center of the loaded area and the loaded grid point being coincident.

In addition to a moment determination for the application of a single concentrated load at the slab center, an attempt will be made to study the conditions that result from expanding the concentrated load until full uniform loading is again achieved.

VI. TWO-WAY SLAB ANALYSIS - UNIFORM LOAD

Design Criteria

(1) The slab is to be two-way reinforced concrete of dimensions 20 ft by 20 ft.

(2) Minimum slab thickness (t) = greater of ACI 318-63, paragraph 2002(e). (3-1/2 inches or $\frac{\text{Slab perimeter}}{180} = \frac{80'}{180} = 0.445$ ft = 5.44 inches. Arbitrarily use t = 7 inches to reduce the deflection.)

(3) Slab dead load = 7/12 ft by 150 pcf = 87.5 psf

Live load =	112.5 psf
Total load =	200.0 psf

Although ACI 318-63 Method 3 indicates that the dead load and live load are combined in the determination of the negative moment at the fixed panel edge, they will be used individually in the negative moment calculation for later comparisons.

(4) Reinforced concrete will be considered as having a Poisson's Ratio (μ) = 0.15.

Boundary Conditions

Case "A"	All edges fixed
Case "B"	Two opposite edges fixed - two opposite edges pinned
Case "C"	All edges pinned

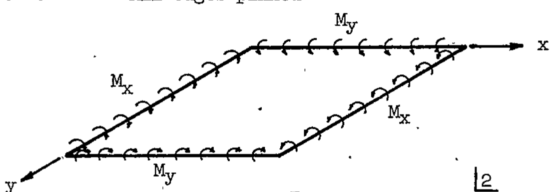


Figure 1.- Positive External Moments.

Design Assumptions

(1) Reinforced concrete is elastic, homogeneous, and isotropic. Although this is a tremendous oversimplification of the exact nature of reinforced concrete, to date, few (if any), of the exact characteristics can be defined. The reason this difficulty exists is due to the vast number of variables that influence its strength. For example, curing temperature, water-cement ratio, strength and gradation of aggregate, placement of reinforcing, placement of forms, workmanship, etc., all control the strength of reinforced concrete to a certain degree. Even though the results obtained will only be an approximation of the exact results, this assumption normally gives a reasonable prediction of the slab action and serves to simplify the analysis.

(2) Plain sections remain plain.

(3) Neutral axis is undeformed.

(4) Deflections are small in comparison to the slab thickness.

(5) The slab is thin in relation to its linear dimensions.

Moment Determination by ACI 318-63 Method 3

Case "A" All edges fixed ACI Case 2

$$m = \frac{X}{Y} = \frac{20'}{20'} = 1.00$$

$$M_x = C_B \cdot q \cdot X^2$$

$$M_y = C_A \cdot q \cdot Y^2$$

where

C = moment coefficient as given in tables 1, 2, and 3

q = uniform load

X and Y = length of respective sides

Positive Moment Calculation

From table 2

$$q_{DL} = D.L. = 87.5 \text{ psf}$$

$$C_{A\ DL} = C_{B\ DL} = 0.018$$

$$M_{x\ DL} = M_{y\ DL} = (0.018)(87.5)(20)^2 = 630 \text{ ft-lb}$$

From table 3

$$q_{LL} = L.L. = 112.5 \text{ psf}$$

$$C_{A\ LL} = C_{B\ LL} = 0.027$$

$$M_{x\ LL} = M_{y\ LL} = (0.027)(112.5)(20)^2 = 1215 \text{ ft-lb}$$

Negative Moment Calculation

From table 1

$$q_{DL} = 87.5 \text{ psf}$$
$$q_{LL} = 112.5 \text{ psf}$$

$$C_{A\ neg} = C_{B\ neg} = -0.045$$

$$M_{x\ DL\ neg} = M_{y\ DL\ neg} = (-0.045)(87.5)(20)^2 = -1575 \text{ ft-lb}$$

$$M_{x\ LL\ neg} = M_{y\ LL\ neg} = (-0.045)(112.5)(20)^2 = -2025 \text{ ft-lb}$$

Summary for Case "A"

$M_{x\ pos} = M_{y\ pos} = M_{y\ DL} + M_{y\ LL} = 630$ ft-lb + 1215 ft-lb = 1845 ft-lb

$M_{x\ neg} = M_{y\ neg} = M_{y\ neg\ DL} + M_{y\ neg\ LL} = -1575$ ft-lb - 2025 ft-lb

$= -3600$ ft-lb

Case "B" — Two opposite edges fixed, two opposite edges pinned — ACI Case 5

$$m = \frac{X}{Y} = 1.0$$

$$M_x = C_B \cdot q \cdot X^2$$

$$M_y = C_A \cdot q \cdot Y^2$$

Positive Moment Calculation

From table 2

$q_{DL} = 87.5$ psf

$$C_{A\ DL} = 0.027$$

$$C_{B\ DL} = 0.018$$

$$M_{x\ DL} = (0.018)(87.5)(20)^2 = 630 \text{ ft-lb}$$

$$M_{y\ DL} = (0.027)(87.5)(20)^2 = 945 \text{ ft-lb}$$

From table 3

$q_{LL} = 112.5$ psf

$$C_{A\ LL} = 0.032$$

$$C_{B\ LL} = 0.027$$

$$M_{x\ LL} = (0.027)(112.5)(20)^2 = 1215 \text{ ft-lb}$$

$$M_{y\ LL} = (0.032)(112.5)(20)^2 = 1440 \text{ ft-lb}$$

Negative Moment Calculation

From table 1

$q_{DL} = 87.5$ psf
$q_{LL} = 112.5$ psf

$$C_{A\ neg} = -0.075$$

$$M_{y\ neg\ DL} = (-0.075)(87.5)(20)^2 = -2625 \text{ ft-lb}$$

$$M_{y\ neg\ LL} = (-0.075)(112.5)(20)^2 = -3375 \text{ ft-lb}$$

Summary for Case "B"

$M_{x\ pos} = M_{x\ DL} + M_{x\ LL} = 630 \text{ ft-lb} + 1215 \text{ ft-lb} = 1845 \text{ ft-lb}$ Middle strip

The ACI Code provides that at discontinuous edges the bending moment in the column strips shall be gradually reduced from the full moment value of the middle strip to 1/3 of these values at the panel edge. Therefore,

$M_{x\ pos\ panel\ edge} = \frac{1}{3}(1845 \text{ ft-lb}) = 615 \text{ ft-lb}$

$M_{y\ pos} = M_{y\ pos\ DL} + M_{y\ pos\ LL} = 945 \text{ ft-lb} + 1440 \text{ ft-lb} = 2385 \text{ ft-lb}$

$M_{y\ neg} = M_{y\ neg\ DL} + M_{y\ neg\ LL} = -2625 \text{ ft-lb} - 3375 \text{ ft-lb} = -6000 \text{ ft-lb}$

Case "C" All edges pinned ACI Case 1

$$m = \frac{X}{Y} = 1.0$$

$$M_x = C_B \cdot q \cdot X^2$$

$$M_y = C_A \cdot q \cdot Y^2$$

Positive Moment Calculation

From table 2

$q_{DL} = 87.5$ psf

$q_{LL} = 112.5$ psf

$$C_A = C_B = 0.036$$

$$M_{x\ DL} = M_{y\ DL} = (0.036)(87.5)(20)^2 = 1260 \text{ ft-lb}$$

$$M_{x\ LL} = M_{y\ LL} = (0.036)(112.5)(20)^2 = 1620 \text{ ft-lb}$$

Summary for Case "C"

$$M_{x\ pos} = M_{y\ pos} = M_{y\ DL} + M_{y\ LL} = 1260 \text{ ft-lb} + 1620 \text{ ft-lb} = 2880 \text{ ft-lb}$$

Middle strip

For the discontinuous edges the moment is 1/3 of the middle strip moment. Therefore,

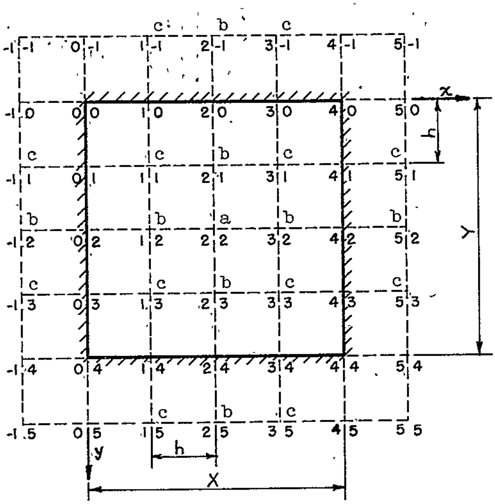
$$M_{x\ pos\ panel\ edge} = M_{y\ pos\ panel\ edge} = \frac{1}{3}(2880) = 960 \text{ ft-lb}$$

There are no negative moments involved in this problem.

Moment Determination by Finite Difference Approximation to the Biharmonic Equation

Finite Difference Approximation to Case "A" (see Appendix)

$$h = \frac{X}{4} = \frac{Y}{4}$$



Boundary Conditions

All edges fixed

Therefore

$$\left.\begin{aligned} w_{-1,y} &= w_{1,y} \\ w_{x,-1} &= w_{x,1} \\ w_{5,y} &= w_{3,y} \\ w_{x,5} &= w_{x,3} \end{aligned}\right\}\ \text{due to rotation restraint}$$

also

$$w_{x,0} = w_{0,y} = w_{x,4} = w_{4,y} = 0$$

due to translation restraint

In applying a uniform load (q) over the entire surface area, conditions of symmetry become apparent. Due to these symmetrical conditions small alphabetic letters will be assigned to points of common deflection, to cut down on writing and make symmetry more obvious.

Superimposing the Finite Difference Approximation to the Biharmonic Equation over each point of differing deflection: Let $C = \frac{q\,h^4}{D}$

At grid point 2,2

$$20a - 8b - 8b - 8b - 8b + 2c + 2c + 2c + 2c = C$$

grouping common terms

$$20a - 32b + 8c = C \tag{1}$$

At grid point 2,1; same for 1,2; 2,3 and 3,2

$$20b - 8a - 8c - 8c + 2b + 2b + b + b = C$$

grouping common terms

$$-8a + 26b - 16c = C \tag{2}$$

At grid point 1,1; same for 1,3; 3,1 and 3,3

$$20c - 8b - 8b + 2a + c + c + c + c = C$$

grouping common terms

$$2a - 16b + 24c = C \tag{3}$$

Placing the simultaneous equations in matrix form

$$\{A \vdots C\} = \begin{array}{c} \\ (3) \\ (2) \\ (1) \end{array} \begin{array}{c} \\ \left[\begin{array}{ccc|c} 2 & -16 & 24 & C \\ -8 & 26 & -16 & C \\ 20 & -32 & 8 & C \end{array}\right] \end{array} \begin{array}{c} \text{Sum} \\ 10+C \\ 2+C \\ -4+C \end{array}$$

The Cholesky or Crout solution yields (see Appendix)

$$L\{T \vdots K\} = \begin{array}{c} \\ (3) \\ (2) \\ (1) \end{array} \begin{array}{c} \\ \left[\begin{array}{ccc|c} 2 & -8 & 12 & .5000C \\ -8 & -38 & -2.1053 & -.1316C \\ 20 & 128 & 37.4784 & .2093C \end{array}\right] \end{array} \begin{array}{c} \text{Check} \\ 5+.5000C \\ -1.1053-.1316C \\ 1.0+.2093C \end{array}$$

Back substitution yields

$c = .2093C$

$b - 2.1053c = - .1316C$

$b = 2.1053(.2093C) - .1316C = .4406C - .1316C$

$b = .3090C$

$a - 8b + 12c = .5C$

$a = 8(.3090C) - 12(.2093C) + .5C = 2.4720C + .5000C - 2.5116C$

$a = .4604C$

Substituting back into original equations

(1) $20(.4604C) - 32(.3090C) + 8(.2093C)$ should $= 1.00C$

$9.2080C - 9.8880C + 1.6744C = + 10.8824C - 9.8880C$

$= .9944C \sim 1.00C$

(2) $-8(.4604C) + 26(.3090C) - 16(.2093C)$ should $= 1.00C$

$-3.6832C + 8.0340C - 3.3488C = + 8.0340C - 7.0320C$

$= 1.0020C \approx 1.00C$

(3) $2(.4604C) - 16(.3090C) + 24(.2093C)$ should $= 1.00C$

$.9208C - 4.9440C + 5.0232C = 5.9440C - 4.9440C$

$= 1.00C$

Therefore the solutions satisfy the original equations fairly well.

The equations for bending moments are [12]

$$M_x = - D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = - D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)$$

The Central Finite Difference Approximation for the second partial derivative is

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{1}{h^2}(w_{i-1,j} \quad -2w_{i,j} \quad w_{i+1,j})$$

or in modular form

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{1}{h^2}\left(1 \quad \overset{\downarrow}{-2} \quad 1 \right)$$

and

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2}\begin{Bmatrix} w_{i,\, j-1} \\ -2\ w_{i,\, j} \\ w_{i,\, j+1} \end{Bmatrix} \text{ or in modular form } \frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2}\begin{Bmatrix} 1 \\ \downarrow \\ -2 \\ \\ 1 \end{Bmatrix}$$

<u>The Maximum Positive Moment</u> occurs at the center of the slab, point 2,2. Therefore superimposing the Central Finite Difference Approximation to the second partial derivatives over point 2,2

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{1}{h^2}(b - 2a + b) = \frac{1}{h^2}(2b - 2a) = \frac{2}{h^2}(.3090 - .4604)C$$

$$= \frac{2C}{h^2}(-\ .1514) = -\ \frac{.3028C}{h^2}$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2}(b - 2a + b) = \frac{1}{h^2}(2b - 2a) = -\ \frac{.3028C}{h^2}$$

$$M_{y\ pos} = M_{x\ pos} \approx -\ D\left(-\ \frac{.3028C}{h^2} + (.15)\left(-\ \frac{.3028C}{h^2}\right)\right)$$

$$3028\ \frac{DC}{h^2}(1.15) = .3482\ \frac{D}{h^2}\ q\ \frac{h^4}{D} = .3482\ qh^2$$

Since

$$q_{DL} = 87.5 \text{ psf} \qquad q_{LL} = 112.5 \text{ psf} \qquad h = \frac{X}{4} = \frac{Y}{4} = \frac{20'}{4} = 5'$$

$$M_{y\ pos\ DL} = M_{x\ pos\ DL} \approx .3482(87.5)(25) = 762 \text{ ft-lb}$$

$$M_{y\ pos\ LL} = M_{x\ pos\ LL} \approx .3482(112.5)(25) = 979 \text{ ft-lb}$$

<u>The Maximum Negative Moments</u> occur at the center of each fixed panel edge, points 0,2; 2,0; 4,2 and 2,4.

$$M_{x\ neg\ 0,2} = M_{x\ neg\ 4,2} = M_{y\ neg\ 2,0} = M_{y\ neg\ 2,4} = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \cancelto{0.}{\frac{\partial^2 w}{\partial x^2}}\right)$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2}(b - 2(0) + b) = \frac{2b}{h^2} = 2\,\frac{(.3090C)}{h^2} = \frac{.6180C}{h^2}$$

Therefore

$$M_{neg} \approx -\frac{D}{h^2}\ .6180\ q\ \frac{h^4}{D} = -\ .6180\ q\ h^2$$

$$M_{x\ neg\ DL} = M_{y\ neg\ DL} \approx -\ .6180(87.5)(25) = -\ 1352 \text{ ft-lb}$$

$$M_{x\ neg\ LL} = M_{y\ neg\ LL} \approx -\ .6180(112.5)(25) = -\ 1738 \text{ ft-lb}$$

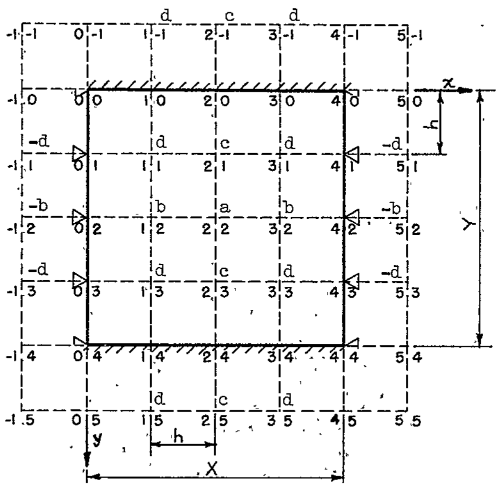
<u>Summary for Case "A"</u>

$$M_{x\ neg} = M_{y\ neg} = M_{y\ neg\ DL} + M_{y\ neg\ LL} \approx -\ 3090 \text{ ft-lb}$$

$$M_{x\ pos} = M_{y\ pos} = M_{y\ pos\ DL} + M_{y\ pos\ LL} \approx 1741 \text{ ft-lb}$$

Finite Difference Approximation to Case "B" (see Appendix)

$$h = \frac{X}{4} = \frac{Y}{4}$$



Boundary Conditions

Two opposite edges fixed, two opposite edges pinned

Therefore

$$\left.\begin{array}{l} w_{-1,y} = -w_{1,y} \\ w_{5,y} = -w_{3,y} \end{array}\right\}$$ due to allowing rotation of pinned edge

$$\left.\begin{array}{l} w_{x,-1} = w_{x,1} \\ w_{x,5} = w_{x,3} \end{array}\right\}$$ due to rotation restraint

$$w_{x,0} = w_{0,y} = w_{x,4} = w_{4,y} = 0$$

due to translation restraint

In applying a uniform load (q) over the entire surface area, conditions of symmetry become apparent. Due to these symmetrical conditions small alphabetic letters will be assigned to points of common deflection, to cut down on writing and make symmetry more obvious.

Superimposing the Finite Difference Approximation to the Biharmonic Equation over each point of differing deflection: Let $C = \frac{q\,h^4}{D}$

At grid point 2,2

$$20a - 8b - 8b - 8c - 8c + 2d + 2d + 2d + 2d = C$$

grouping common terms

$$20a - 16b - 16c + 8d = C \qquad (1)$$

At grid point 1,2; same for 3,2

$$20b - 8a - 8d - 8d + 2c + 2c + b - b = C$$

grouping common terms

$$-8a + 20b + 4c - 16d = C \qquad (2)$$

At grid point 2,3; same for 2,1

$$20c - 8a - 8d - 8d + 2b + 2b + c + c = C$$

grouping common terms

$$-8a + 4b + 22c - 16d = C \qquad (3)$$

At grid point 1,3; same for 1,1; 3,1 and 3,3

$$20d - 8b - 8c + 2a + d + d + d - d = C$$

grouping common terms

$$2a - 8b - 8c + 22d = C \qquad (4)$$

Placing the simultaneous equations in matrix form

$$\{A \vdots C\} = \begin{array}{c} (4) \\ (2) \\ (3) \\ (1) \end{array} \left[\begin{array}{cccc:c} 2 & -8 & -8 & 22 & C \\ -8 & 20 & 4 & -16 & C \\ -8 & 4 & 22 & -16 & C \\ 20 & -16 & -16 & 8 & C \end{array}\right] \begin{array}{:c:} \text{Sum} \\ 8+C \\ C \\ 2+C \\ -4+C \end{array}$$

The Cholesky or Crout solution yields (see Appendix)

$$L\{T \vdots K\} = \left[\begin{array}{rrrr:r} 2 & -4 & -4 & 11 & .5000C \\ -8 & -12 & 2.3333 & -6.0000 & -.4167C \\ -8 & -28 & 55.3324 & -1.7350 & -.1205C \\ 20 & 64 & -85.3312 & 23.9504 & .3084C \end{array}\right] \begin{array}{:l:} \text{Check} \\ 4+.5000C \\ -2.6667-.4167C \\ -.7350-.1205C \\ 1.0+.3084C \end{array}$$

Back substitution yields

$d = .3084C$

$c - 1.7350d = -.1205C$

$c = 1.7350(.3084C) - .1205C = .5351C - .1205C$

$c = .4146C$

$b + 2.3333c - 6.0d = -.4167C$

$b = -2.3333(.4146C) + 6.0(.3084C) - .4167C = 1.8504C - 1.3841C$

$b = .4663C$

$a - 4b - 4c + 11d = .5000C$

$a = 4(.4663C) + 4(.4146C) - 11(.3084C) + .5000C$

$a = 4.0236C - 3.3924C$

$a = .6312C$

Substituting back into original equations

(1) $20(.6312C) - 16(.4663C) - 16(.4146C) + 8(.3084C)$ should $= 1.00C$

$$12.6240C - 7.4608C - 6.6336C + 2.4672C = 15.0912C - 14.0944C$$
$$= .9968C \sim 1.00C$$

(2) $-8(.6312C) + 20(.4663C) + 4(.4146C) - 16(.3084C)$ should $= 1.00C$

$$-5.0496C + 9.3260C + 1.6584C - 4.9344C = 10.9844C - 9.9840C$$
$$= 1.0004C \sim 1.00C$$

(3) $-8(.6312C) + 4(.4663C) + 22(.4146C) - 16(.3084C)$ should $= 1.00C$

$$-5.0496C + 1.8652C + 9.1212C - 4.9344C = 10.9864C - 9.9840C$$

$$= 1.0024C \sim 1.00C$$

(4) $2(.6312C) - 8(.4663C) - 8(.4146C) + 22(.3084C)$ should $= 1.00C$

$$1.2624C - 3.7304C - 3.3168C + 6.7848C = 8.0472C - 7.0472C$$

$$= 1.0000C$$

Therefore the solutions satisfy the original equations fairly well.

The equations for bending moments are (see F.D. Approx. for Case "A")

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)$$

<u>The Maximum Positive Moments</u> occur at the center of the slab, point 2,2.

Therefore, superimposing the Central Finite Difference Approximation to the second partial derivatives over point 2,2

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{1}{h^2}(b - 2a + b) = \frac{1}{h^2}(2b - 2a) = \frac{2}{h^2}(.4663 - .6312)C$$

$$= \frac{2C}{h^2}(-.1649) = -\frac{.3298C}{h^2}$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2}(c - 2a + c) = \frac{1}{h^2}(2c - 2a) = \frac{2}{h^2}(.4146 - .6312)C$$

$$= \frac{2C}{h^2}(-.2166) = -\frac{.4332C}{h^2}$$

$$M_{x\ pos} \approx -D\left(-\frac{.3298C}{h^2} + (.15)\left(-\frac{.4332C}{h^2}\right)\right)$$

$$\approx -\frac{DC}{h^2}(-.3298 - .0650) = .3948\frac{D}{h^2}q\frac{h^4}{D}$$

$$\approx .3948\ q\ h^2$$

$$M_{y\ pos} \approx -D\left(-\frac{.4332C}{h^2} + (.15)\left(-\frac{.3298C}{h^2}\right)\right) = -\frac{DC}{h^2}(-.4332 - .0495)$$

$$\approx .4827\frac{D}{h^2}q\frac{h^4}{D} = .4827\ q\ h^2$$

Since

$q_{DL} = 87.5$ psf $\quad q_{LL} = 112.5$ psf $\quad h = \frac{X}{4} = \frac{Y}{4} = \frac{20'}{4} = 5'$

$$M_{x\ pos\ DL} \approx .3948(87.5)(25) = 864 \text{ ft-lb}$$

$$M_{x\ pos\ LL} \approx .3948(112.5)(25) = 1110 \text{ ft-lb}$$

$$M_{y\ pos\ DL} \approx .4827(87.5)(25) = 1056 \text{ ft-lb}$$

$$M_{y\ pos\ LL} \approx .4827(112.5)(25) = 1358 \text{ ft-lb}$$

<u>The Maximum Negative Moments</u> occur at the center of each fixed panel edge, points 2,0 and 2,4.

$$M_{y\ neg}' = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\,\cancelto{0}{\frac{\partial^2 w}{\partial x^2}}\right)$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2}(c - 2(0) + c) = \frac{2c}{h^2} = \frac{2}{h^2}(.4146C)$$

$$\approx .8292 \frac{C}{h^2}$$

$$M_{y\ neg} \approx - D\left(.8292 \frac{C}{h^2}\right) = - .8292 \frac{D}{h^2} q \frac{h^4}{D} = - .8292\ q\ h^2$$

$$M_{y\ neg\ DL} \approx - .8292(87.5)(25) = - 1814 \text{ ft-lb}$$

$$M_{y\ neg\ LL} \approx - .8292(112.5)(25) \doteq - 2332 \text{ ft-lb}$$

Summary for Case "B"

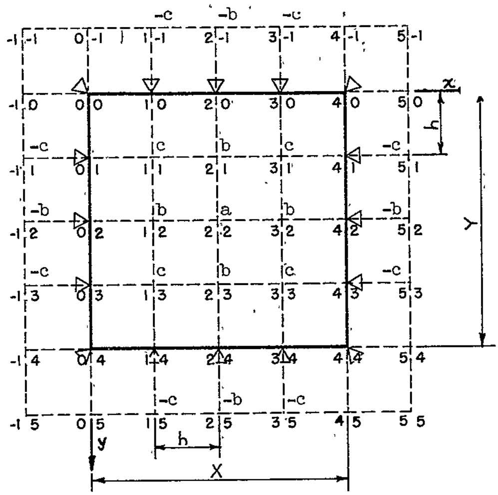
$$M_{y\ neg} = M_{y\ neg\ DL} + M_{y\ neg\ LL} \approx - 4146 \text{ ft-lb}$$

$$M_{x\ pos} = M_{x\ pos\ DL} + M_{x\ pos\ LL} \approx 1974 \text{ ft-lb}$$

$$M_{y\ pos} = M_{y\ pos\ DL} + M_{y\ pos\ LL} \approx 2414 \text{ ft-lb}$$

<u>Finite Difference Approximation to Case "C"</u> (see Appendix)

$$h = \frac{X}{4} = \frac{Y}{4}$$



<u>Boundary Conditions</u>

All edges pinned

Therefore

$$\left.\begin{aligned} w_{-1,y} &= -w_{1,y} \\ w_{x,-1} &= -w_{x,1} \\ w_{5,y} &= -w_{3,y} \\ w_{x,5} &= -w_{x,3} \end{aligned}\right\}$$ due to allowing rotation of pinned edge

$$w_{x,0} = w_{0,y} = w_{x,4} = w_{4,y} = 0$$

due to translation restraint

In applying a uniform load (q) over the entire surface area, conditions of symmetry become apparent. Due to these symmetrical conditions small alphabetic letters will be assigned to points of common deflection, to cut down on writing and make symmetry more obvious.

Superimposing the Finite Difference Approximation to the Biharmonic Equation over each point of differing deflection: Let $C = q\,\frac{h^4}{D}$

At grid point 2,2

$$20a - 8b - 8b - 8b - 8b + 2c + 2c + 2c + 2c = C$$

grouping common terms

$$20a - 32b + 8c = C \qquad (1)$$

At grid point 2,1; same for 1,2; 2,3 and 3,2

$$20b - 8a - 8c - 8c + 2b + 2b + b - b = C$$

grouping common terms

$$-8a + 24b - 16c = C \tag{2}$$

At grid point 1,1; same for 1,3; 3,1 and 3,3

$$20c - 8b - 8b + 2a + c + c - c - c = C$$

grouping common terms

$$2a - 16b + 20c = C \tag{3}$$

Placing the simultaneous equations in matrix form

$$\{A \vdots C\} = \begin{array}{c} (3) \\ (2) \\ (1) \end{array} \left[\begin{array}{ccc:c} 2 & -16 & 20 & C \\ -8 & 20 & -16 & C \\ 20 & -32 & 8 & C \end{array}\right] \begin{array}{c} \text{Sum} \\ 6+C \\ C \\ -4+C \end{array}$$

The Cholesky or Crout solution yields (see Appendix)

$$L\{T \vdots K\} = \left[\begin{array}{ccc:c} 2 & -8 & 10 & .5000C \\ -8 & -40 & -1.6000 & -.1250C \\ 20 & 128 & 12.8000 & .5469C \end{array}\right] \begin{array}{c} \text{Check} \\ 3+.5000C \\ -.6000-.1250C \\ 1.0+.5469C \end{array}$$

Back substitution yields

$$c = .5469C$$

$$b - 1.6000c = -.1250C$$

$$b = 1.6000(.5469C) - .1250C = .8750C - .1250C$$

$$b = .7500C$$

$$a - 8b + 10c = .5000C$$

$$a = 8(.7500C) - 10(.5469C) + .5000C = 6.5000C - 5.4690C$$

$$a = 1.0310C$$

Substituting back into original equations

(1) $20(1.0310C) - 32(.7500C) + 8(.5469C)$ should $= 1.00C$

$$20.6200C - 24.0000C + 4.3752C = 24.9952C - 24.0000C$$

$$= .9952C \sim 1.00C$$

(2) $-8(1.0310C) + 24(.7500C) - 16(.5469C)$ should $= 1.00C$

$$-8.2480C + 18.0000C - 8.7504C = 18.0000C - 16.9984C$$

$$= 1.0016C \sim 1.00C$$

(3) $2(1.0310C) - 16(.7500C) + 20(.5469C)$ should $= 1.00C$

$$2.0620C - 12.0000C + 10.9380C = 13.0000C - 12.0000C$$

$$= 1.0000C$$

Therefore the solutions satisfy the original equations fairly well.

The equations for bending moments are (see F.D. Approx. for Case "A")

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)$$

<u>The Maximum Positive Moments</u> occur at the center of the slab, point 2,2. Therefore superimposing the Central Finite Difference Approximation to the second partial derivatives over point 2,2

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{1}{h^2}(b - 2a + b) = \frac{1}{h^2}(2b - 2a) = \frac{2}{h^2}(.7500 - 1.0310)C$$

$$= \frac{2C}{h^2}(-.2810) = -.5620\frac{C}{h^2}$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2}(b - 2a + b) = \frac{1}{h^2}(2b - 2a) = -\ .5620\ \frac{C}{h^2}$$

$$M_{y\ pos} = M_{x\ pos} \approx -\ D\left(-\ .5620\ \frac{C}{h^2} + (.15)\left(-\ .5620\ \frac{C}{h^2}\right)\right)$$

$$\approx .5620\ \frac{DC}{h^2}(1.15) = .6463\ \frac{D}{h^2}\ q\ \frac{h^4}{D}$$

$$\approx .6463\ q\ h^2$$

Since

$$q_{DL} = 87.5\ \text{psf} \qquad q_{LL} = 112.5\ \text{psf} \qquad h = \frac{X}{4} = \frac{Y}{4} = \frac{20'}{4} = 5'$$

$$M_{y\ pos\ DL} = M_{x\ pos\ DL} \approx .6463(87.5)(25) = 1414\ \text{ft-lb}$$

$$M_{y\ pos\ LL} = M_{x\ pos\ LL} \approx .6463(112.5)(25) = 1818\ \text{ft-lb}$$

<u>There is no negative moment involved in this case.</u>

<u>Summary for Case "C"</u>

$$M_{neg} = 0$$

$$M_{y\ pos} = M_{x\ pos} = M_{x\ pos\ DL} + M_{x\ pos\ LL} = 3232\ \text{ft-lb}$$

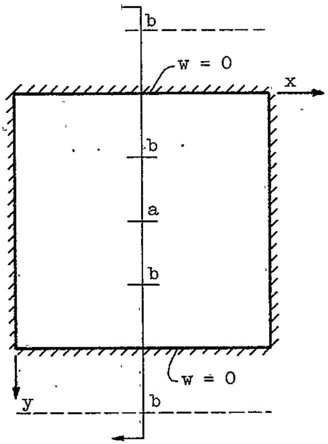
Comparison of Results for Uniform Loading Condition

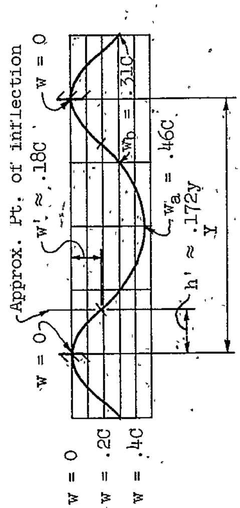
Maximum Moment	Grid Point	ACI 318-63 Method 3, ft-lb	Central Finite Difference Approximation, ft-lb	Finite Difference Approximation, percent of ACI	Elastic Curve Plot ft-lb
Case "A"					
M_x	2,2	+1845	+1741	94.4	
M_y	2,2	+1845	+1741	94.4	
M_x	$\begin{cases}0,2\\4,2\end{cases}$	-3600	-3090	85.8	-3800
M_y	$\begin{cases}2,0\\2,4\end{cases}$				
Case "B"					
M_x	$\begin{cases}0,2\\4,2\end{cases}$	+615	0		
M_x	2,2	+1845	+1974	107.0	
M_y	2,2	+2385	+2414	101.2	
M_y	$\begin{cases}2,0\\2,4\end{cases}$	-6000	-4146	69	-5587
Case "C"					
M_x	$\begin{cases}0,2\\4,2\end{cases}$	+960	0		
M_y	$\begin{cases}2,0\\2,4\end{cases}$	+960	0		
M_x	2,2	+2880	+3232	112.2	
M_y	2,2	+2880	+3232	112.2	

Table 1. Design Moments for Uniform Loading Condition

The results compare quite favorably with the exception of the maximum fixed edge moments in both Case "A" and Case "B". This indicates that the grid chosen was too large to closely approximate the deflections that influence the maximum negative moment. However, since halving the grid intervals approximately quadruples the amount of work necessary to obtain the solutions to the simultaneous equations, it would appear that a more simple method of approximating these controlling deflections is in order. This can be done by sketching the elastic curve of the slab section through the point of interest, using the calculated deflections. The inflection point in the elastic curve is then approximately located and the deflection and distance from the point under consideration is scaled off the curve. (Note - a thin spring steel wire with straight pins at plotted deflection points works wonders in approximating the elastic curve). The second partial derivative is then recalculated using the new values obtained and the moment is modified accordingly.

<u>Elastic Curve Plot for Case "A" Negative Moment</u>





From elastic curve

$$h' \approx .172Y$$

$$w' \approx .18C$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{(h')^2}(2w') = \frac{2(.18C)}{(.172Y)^2} = \frac{12.17C}{Y^2}$$

Therefore

$$M_{x\ neg} = M_{y\ neg} = -D\left(\frac{\partial^2 w}{\partial y^2}\right) \approx -D\left(\frac{12.17C}{Y^2}\right)$$

with

$$Y = 20', \quad h = \frac{Y}{4}, \quad C = q\,\frac{h^4}{D} = q\,\frac{Y^4}{256D}$$

Since

$q_{DL} = 87.5$ psf . . $\qquad q_{LL} = 112.5$ psf

$$M_{x\&y\ neg\ DL} \approx -D\frac{(12.17)}{Y^2}\,\frac{(87.5)Y^4}{(256)D} = -\frac{(12.17)(87.5)Y^2}{256}$$

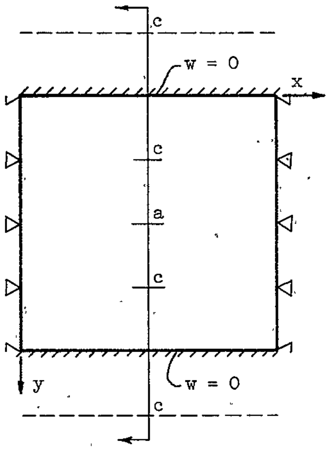
$$\approx -.0475(87.5)(400) = -1662\ \text{ft-lb}$$

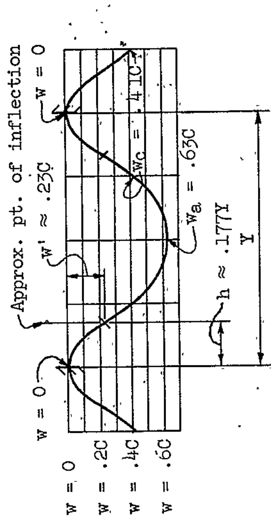
$$M_{x\&y\ neg\ LL} \approx -D\,\frac{(12.17)}{Y^2}\,\frac{(112.5)Y^4}{(256)D} = -\frac{(12.17)(112.5)Y^2}{256}$$

$$\approx -.0475(112.5)(400) = -2138\ \text{ft-lb}$$

$$M_{x\&y\ neg} \doteq M_{y\ neg\ DL} + M_{y\ neg\ LL} = -3800\ \text{ft-lb}$$

Elastic Curve Plot for Case "B" Negative Moment





From elastic curve

$$h' \approx .177Y$$

$$w' \approx .23C$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{(h')^2}(2w') = \frac{2(.23C)}{(.177Y)^2} = \frac{17.88C}{Y^2}$$

Therefore

$$M_{y\,neg} = -D\left(\frac{\partial^2 w}{\partial y^2}\right) \approx -D\left(\frac{17.88C}{Y^2}\right)$$

with

$$Y = 20', \quad h = \frac{Y}{4}, \quad C = q\,\frac{h^4}{D} = q\,\frac{Y^4}{256D}$$

Since

$$q_{DL} = 87.5 \text{ psf} \qquad\qquad q_{LL} = 112.5 \text{ psf}$$

$$M_{y \text{ neg } DL} \approx -D\,\frac{(17.88)}{Y^2}\,\frac{(87.5)Y^4}{(256)D} = -.06984(87.5)Y^2$$

$$\approx -.06984(87.5)(400) = -2444 \text{ ft-lb}$$

$$M_{y \text{ neg } LL} \approx -D\,\frac{(17.88)}{Y^2}\,\frac{(112.5)Y^4}{256D} = -.06984(112.5)Y^2$$

$$\approx -.06984(112.5)(400) = -3143 \text{ ft-lb}$$

$$M_{y \text{ neg}} = M_{y \text{ neg } DL} + M_{y \text{ neg } LL} \approx -5587 \text{ ft-lb}$$

VII. TWO-WAY SLAB ANALYSIS - CONCENTRATED LOADS

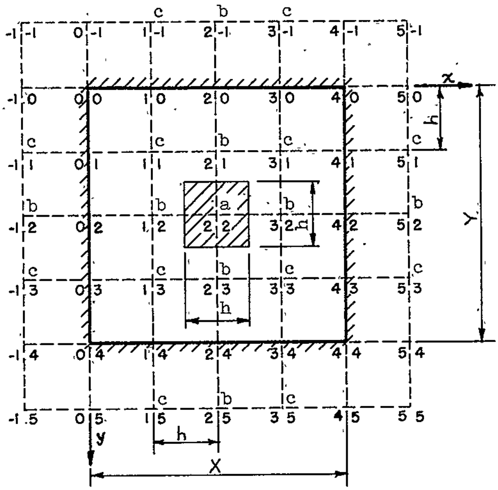
An examination of the preceding Table 1 indicates that the results obtained by the Central Finite Difference Approximation to the Biharmonic Equation along with the Elastic Curve Plot compares well with the results obtained by the analytical, emperical or experimental methods used to establish Method 3 of ACI 318-63. Therefore, the Finite Difference Approximation can be used to analyze a slab for the application of concentrated loads and a reasonable degree of accuracy can be expected.

This study will now explore the application of concentrated loads on the three cases previously examined. It will attempt to investigate how the maximum moments are affected as the load is expanded from application over an area $\frac{X}{4}$ by $\frac{Y}{4}$ to full uniform loading. The full live load $(q \cdot X \cdot Y)$ applied to the slab under the uniform loading condition will still be used, only it will be applied over the smaller area. For this study, the load will be located such that the symmetry of the deflections is maintained. The assumptions previously stated still apply.

Finite Difference Approximation to Case "A"

Concentrated Load Over Grid Point 2,2 (see Appendix)

$h = \frac{X}{4} = \frac{Y}{4}$



The total live load previously placed on the slab will now be concentrated over an area $h \times h$ at point 2,2.

Boundary Conditions

All edges fixed ;

Therefore

$$\left.\begin{aligned} w_{-1,y} &= w_{1,y} \\ w_{x,-1} &= w_{x,1} \\ w_{5,y} &= w_{3,y} \\ w_{x,5} &= w_{x,3} \end{aligned}\right\} \text{due to rotation restraint}$$

also

$w_{x,0} = w_{0,y} = w_{x,4} = w_{4,y} = 0$

due to translation restraint

Since the deflection symmetry conditions are retained by the load placement, the same small alphabetic letters can be assigned to the points of common deflection.

Previous total live load = 112.5 psf × 20' × 20' = 45000 lb = W

$$q_a = \frac{W}{h^2}.$$

Superimposing the Finite Difference Approximation to the Biharmonic Equation over each point of differing deflection: Let $R = q_a \frac{h^4}{D} = \frac{Wh^2}{D}$

At grid point 2,2

$$20a - 8b - 8b - 8b - 8b + 2c + 2c + 2c + 2c = R$$

grouping common terms

$$20a - 32b + 8c = R \tag{1}$$

At grid point 2,1; same for 1,2; 2,3 and 3,2

$$20b - 8a - 8c - 8c + 2b + 2b + b + b = 0$$

grouping common terms

$$-8a + 26b - 16c = 0 \tag{2}$$

At grid point 1,1; same for 1,3; 3,1 and 3,3

$$20c - 8b - 8b + 2a + c + c + c + c = 0$$

grouping common terms

$$2a - 16b + 24c = 0 \tag{3}$$

Placing the simultaneous equations in matrix form

$$\left\{A \vdots C\right\} = \begin{array}{c} (3) \\ (2) \\ (1) \end{array} \left[\begin{array}{ccc:c} 2 & -16 & 24 & 0 \\ -8 & 26 & -16 & 0 \\ 20 & -32 & 8 & R \end{array}\right] \begin{array}{c} \text{Sum} \\ \vdots\ 10 \\ \vdots\ 2 \\ \vdots\ -4+R \end{array}$$

It should be noted that a similartiy exists between the preceding simultaneous equations and those obtained under the uniform loading consideration. The only difference is in the value of the load placed on the grid points. Therefore, a large portion of the. $L\left\{T \vdots K\right\}$ matrix remains the same.

$$L\{T \vdots K\} = \left[\begin{array}{ccc:c} 2 & -8 & 12 & 0 \\ -8 & -38 & -2.1053 & 0 \\ 20 & 120 & 37.4784 & .0267R \end{array}\right] \vdots \begin{array}{c} \text{Check} \\ 5 \\ -1.1053 \\ 1.0+.0267R \end{array}$$

Back substituting yields

$c = .0267R$

$b - 2.1053c = 0$

$b = 2.1053(.0267R)$

$b = .0562R$

$a - 8b + 12c = 0$

$a = 8(.0562R) - 12(.0267R) = .4497R - .3204R$

$a = .1293R$

Substituting back into original equations

(1) $20(.1293R) - 32(.0562R) + 8\,(.0267R)$ should $= 1.00R$

$2.5860R - 1.7987R - .2136R = 2.7996R - 1.7987R$

$= 1.0009R \sim 1.00R$

(2) $-8(.1293R) + 26(.0562R) - 16(.0267R)$ should $= 0$

$-1.0344R + 1.4615R - .4272R = 1.4615R - 1.4616R$

$= -.0001R \sim 0$

(3) $2(.1293R) - 16(.0562R) + 24(.0267R)$ should $= 0$

$.2586R - .8992R + .6408R = .8994R - .8992R$

$= .0002R \sim 0$

Therefore the solutions satisfy the original equations fairly well.

The equations for bending moments are (see F.D. Approx. for Case "A" uniform load)

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right)$$

<u>The Maximum Positive Moments</u> occur at the center of the slab, point 2,2. Therefore superimposing the Central Finite Difference Approximation to the second partial derivatives over point 2,2

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{1}{h^2}(b - 2a + b) = \frac{1}{h^2}(2b - 2a) = \frac{2}{h^2}(.0562 - .1293)R$$

$$= \frac{2R}{h^2}(-.0731) = -.1462\frac{R}{h^2}$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2}(b - 2a + b) = -.1462\frac{R}{h^2}$$

$$M_y \text{ pos LL} = M_x \text{ pos LL} \approx -D\left(-.1462\frac{R}{h^2} + (.15)\left(-.1462\frac{R}{h^2}\right)\right)$$

$$\approx .1462\frac{RD}{h^2}(1.15) = .1681\frac{D}{h^2}R$$

$$\approx .1681\frac{D}{h^2}\frac{Wh^2}{D} = .1681W$$

$$\approx .1681(45000)$$

$$\approx 7565 \text{ ft-lb}$$

<u>The Maximum Negative Moments</u> occur at the center of each fixed panel edge, points 0,2; 2,0; 4,2 and 2,4. Although the Finite Difference Approximation analysis for the uniform loading condition indicated the grid spacing to be too coarse to closely approximate the maximum negative moments, the negative moments will be calculated for an order of magnitude determination and then checked with the Elastic Curve Plot.

$$M_{x\ neg\ 0,2} = M_{x\ neg\ 4,2} = M_{y\ neg\ 2,0} = M_{y\ neg\ 2,4} = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \cancelto{0}{\frac{\partial^2 w}{\partial x^2}}\right)$$

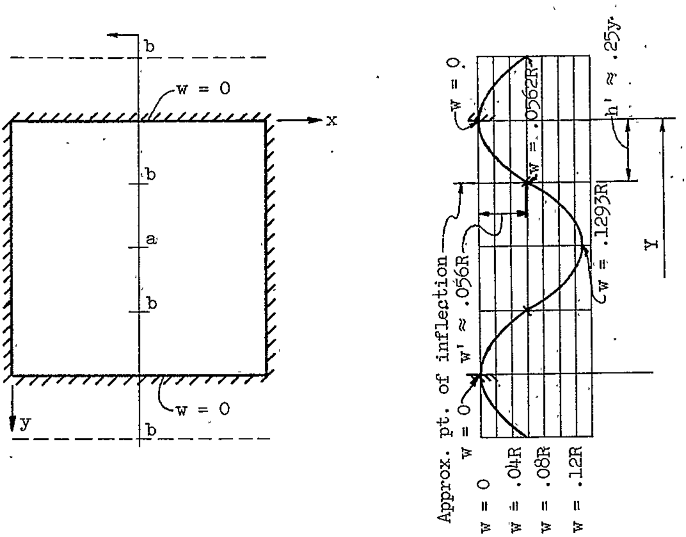
$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2}(b - 2(0) + b) = \frac{2b}{h^2} = \frac{2}{h^2}(.0562R) = \frac{.1124R}{h^2}$$

$$M_{x\&y\ neg\ LL} \approx -D\left(\frac{.1124R}{h^2}\right) = -.1142\frac{D}{h^2}\frac{Wh^2}{D} = -.1142W$$

$$\approx -.1142(45000)$$

$$\approx -5139 \text{ ft-lb}$$

Elastic Curve Plot for Negative Moment.

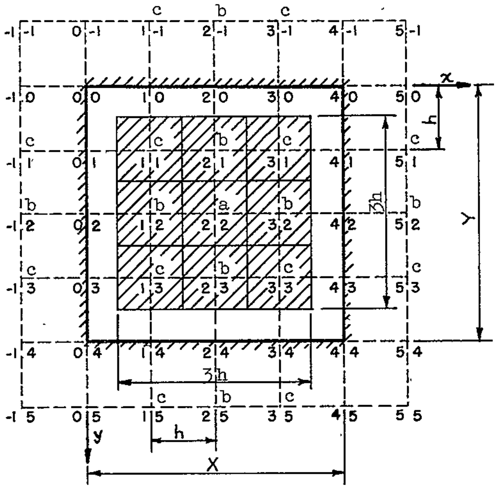


The Elastic Curve Plot indicates that the inflection point is very near the grid point. Therefore the negative moment calculation is approximately correct.

Finite Difference Approximation to Case "A" (see Appendix)

Concentrated Load Over all Internal Grid Points

$$h = \frac{X}{4} = \frac{Y}{4}$$



Boundary Conditions

All edges fixed

Therefore

$$\left.\begin{aligned} w_{-1,y} &= w_{1,y} \\ w_{x,-1} &= w_{x,1} \\ w_{5,y} &= w_{3,y} \\ w_{x,5} &= w_{x,3} \end{aligned}\right\} \text{due to rotation restraint}$$

also

$$w_{x,0} = w_{0,y} = w_{x,4} = w_{4,y} = 0$$

due to translation restraint

Since the deflection symmetry conditions are retained by the load placement, the same small alphabetic letters can be assigned to the points of common deflection.

The total live load previously placed on the slab will now be distributed over an area $3h \times 3h$, effectively concentrating a portion of the load over each internal grid point.

Previous total live load = 112.5 psf × 20' × 20' = 45000 lb = W

$$q_t = \frac{W}{(3h)^2} = \frac{45000}{9h^2} = \frac{5000}{h^2}$$

Superimposing the Finite Difference Approximation to the Biharmonic Equation over each point of differing deflection: Let $P = q_t \dfrac{h^4}{D} = 5000 \dfrac{h^2}{D}$.

At grid point 2,2

$$20a - 8b - 8b - 8b - 8b + 2c + 2c + 2c + 2c = P$$

grouping common terms

$$20a - 32b + 8c = P \tag{1}$$

At grid point 2,1; same for 1,2; 2,3 and 3,2

$$20b - 8a - 8c - 8c + 2b + 2b + b + b = P$$

grouping common terms

$$-8a + 26B - 16c = P \tag{2}$$

At grid point 1,1; same for 1,3; 3,1 and 3,3

$$20c - 8b - 8b + 2a + c + c + c + c = P$$

grouping common terms

$$2a - 16b + 24c = P \tag{3}$$

Placing the simultaneous equations in matrix form

$$\{A \vdots C\} = \begin{array}{c} \\ \left[\begin{array}{ccc:c} & -16 & 24 & P \\ -8 & 26 & -16 & P \\ 20 & -32 & 8 & P \end{array}\right] \end{array} \begin{array}{c} \text{Sum} \\ \begin{array}{c} 10+P \\ 2+P \\ -4+P \end{array} \end{array}$$

Note that the $\{A \vdots C\}$ matrix above is identical with that of the original uniform loading consideration. Therefore the solutions to the simultaneous equations and the deflection coefficients are identical. The difference in the moments will come from the different load applied at the grid points.

Therefore

$a = .4604P$

$b = .3090P$

$c = .2093P$

The equations for bending moments are (see F.D. Approx. for Case "A", uniform load)

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)$$

The Maximum Positive Moments occur at the center of the slab, point 2,2.

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} \approx -\frac{.3028P}{h^2}$$

$$M_{y \text{ pos LL}} = M_{x \text{ pos LL}} \approx -D\left(-\frac{.3028P}{h^2} + (.15)\left(-.3028\frac{P}{h^2}\right)\right)$$

$$\approx .3482\frac{DP}{h^2} = .3482\frac{D}{h^2}\left(5000\frac{h^2}{D}\right)$$

$$\approx 1741 \text{ ft-lb}$$

The Maximum Negative Moments occur at the center of each fixed panel edge, points 0,2; 2,0; 4,2 and 2,4. Although the Finite Difference Approximation analysis for the uniform loading condition indicated the grid spacing to be too coarse to closely approximate the maximum negative moments, the negative moments will be calculated for an order of magnitude determination and then checked with the Elastic Curve Plot.

$$M_x \text{ neg } 0,2 = M_x \text{ neg } 4,2 = M_y \text{ neg } 2,0 = M_y \text{ neg } 2,4 = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \cancelto{0}{\frac{\partial^2 w}{\partial x^2}}\right)$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{.6180P}{h^2}$$

$$M_{x\&y} \text{ neg LL} \approx -D\left(\frac{.6180P}{h^2}\right) = -\ .6180\ \frac{D}{h^2}\left(5000\ \frac{h^2}{D}\right)$$

$$3090 \text{ ft-lb}$$

The Elastic Curve Plot for this loading condition is the same as that previously shown for Case "A" uniform load. Therefore, the curve data at the inflection point is:

$$h' \approx .172Y$$

$$w' \approx .18P$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{12.17P}{Y^2}$$

Therefore

$$M_{x\&y} \text{ neg LL} = -D\left(\frac{\partial^2 w}{\partial y^2}\right) \approx -D\left(\frac{12.17P}{Y^2}\right)$$

with

$$h = \frac{Y}{4},$$

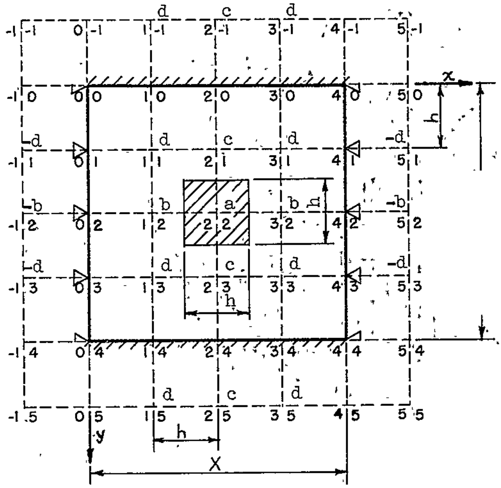
$$P = \frac{5000h^2}{D} = \frac{5000Y^2}{16D}$$

$$M_{x\&y} \text{ neg LL} \approx -D\left(\frac{12.17}{Y^2}\right)\left(\frac{5000Y^2}{16D}\right) = -\ 3803 \text{ ft-lb}$$

Finite Difference Approximation to Case "B"

Concentrated Load Over Grid Point 2,2 (see Appendix)

$$h = \frac{X}{4} = \frac{Y}{4}$$



The total live load previously placed on the slab will now be concentrated over an area h × h at point 2,2.

Boundary Conditions

Two opposite edges fixed, two opposite edges pinned

Therefore

$$\left.\begin{aligned} w_{-1,y} &= -w_{1,y} \\ w_{5,y} &= -w_{3,y} \end{aligned}\right\} \text{due to allowing rotation of pinned edge}$$

$$\left.\begin{aligned} w_{x,-1} &= w_{x,1} \\ w_{x,5} &= w_{x,3} \end{aligned}\right\} \text{due to rotation restraint}$$

$w_{x,0} = w_{0,y} = w_{x,4} = w_{4,y} = 0$ due to translation restraint

Since the deflection symmetry conditions are retained by the load placement, the same small alphabetic letters can be assigned to the points of common deflection.

Previous total live load = 112.5 psf × 20' × 20' = 45000 lb = W

$$q_a = \frac{W}{h^2}$$

Superimposing the Finite Difference Approximation to the Biharmonic Equation over each point of differing deflection: Let $R = q_a \frac{h^4}{D} = \frac{Wh^2}{D}$

At grid point 2,2

$$20a - 8b - 8b - 8c - 8c + 2d + 2d + 2d + 2d \doteq R$$

grouping common terms

$$20a - 16b - 16c + 8d = R \tag{1}$$

At grid point 1,2; same for 3,2

$$20b - 8a - 8d - 8d + 2c + 2c + b - b = 0$$

grouping common terms

$$-8a + 20b + 4c - 16d = 0 \tag{2}$$

At grid point 2,3; same for 2,1

$$20c - 8a - 8d - 8d + 2b + 2b + c + c = 0$$

grouping common terms

$$-8a + 4b + 22c - 16d = 0 \tag{3}$$

At grid point 1,3; same for 1,1; 3,1 and 3,3

$$20d - 8b - 8c + 2a + d + d + d - d = 0$$

grouping common terms

$$2a - 8b - 8c + 22d = 0 \tag{4}$$

Placing the simultaneous equations in matrix form

$$\{A : C\} = \begin{array}{c} (4) \\ (2) \\ (3) \\ (1) \end{array} \left[\begin{array}{cccc:c} 2 & -8 & -8 & 22 & 0 \\ -8 & 20 & 4 & -16 & 0 \\ -8 & 4 & 22 & -16 & 0 \\ 20 & -16 & -16 & 8 & R \end{array}\right] \begin{array}{:c:} 0 \\ 0 \\ 2 \\ -4+R \end{array}$$

It should be noted that a similarity exists between the preceding simultaneous equations and those obtained under the uniform loading consideration. The only difference is in the value of the load placed on the grid points. Therefore, a large portion of the $L\{T:K\}$ matrix remains the same.

$$L\{T : K\} = \begin{bmatrix} 2 & -4 & -4 & 11 & \vdots & 0 \\ -8 & -12 & 2.3333 & -6.0000 & \vdots & 0 \\ -8 & -28 & 55.3324 & -1.7350 & \vdots & 0 \\ 20 & 64 & -85.3312 & 23.9504 & \vdots & .0418R \end{bmatrix} \begin{array}{c} \text{Check} \\ \vdots\ 4 \\ \vdots\ -2.6667 \\ \vdots\ -.7350 \\ \vdots\ 1.0+.0418R \end{array}$$

Back substituting yields

$d = .0418R$

$c - 1.7350d = 0$

$c = 1.7350(.0418R)$

$c = .0725R$

$b + 2.3333c - 6.0000d = 0$

$b = -2.333(.0725R) + 6.000(.0418R)$

$b = .0816R$

$a - 4b - 4c + 11d = 0$

$a = 4(.0816R) + 4(.0725R) - 11(.0418R)$

$a = .1566R$

Substituting back into original equations

(1) $20(.1566R) - 16(.0816R) - 16(.0725R) + 8(.0418R)$ should $= 1.00R$

$$3.1320R - 1.3056R - 1.1600R + .3344R = 3.4664R - 2.4656R$$

$$= 1.0008R \sim 1.00R$$

(2) $-8(.1566R) + 20(.0816R) + 4(.0725R) - 16(.0418R)$ should $= 0$

$$-1.2528R + 1.6320R + .2900R - .6688R = 1.9220R - 1.9216R$$

$$= .0004R \sim 0$$

(3) $-8(.1566R) + 4(.0816R) + 22(.0725R) - 16(.0418R)$ should $= 0$

$$-1.2528R + .3264R + 1.5950R - .6688R = 1.9214R - 1.9216R$$

$$= -.0002R \sim 0$$

(4) $2(.1566R) - 8(.0816R) - 8(.0725R) + 22(.0418R)$ should $= 0$

$$.3132R - .6528R - .5800R + .9196R = 1.2328R - 1.2328R$$

$$= 0.0000R$$

Therefore the solutions satisfy the original equations fairly well.

The equations for bending moments are (see F. D. Approx. for Case "A" uniform load)

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right)$$

The Maximum Positive Moments occur at the center of the slab, point 2,2. Therefore superimposing the Central Finite Difference Approximation to the second partial derivatives over point 2,2

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{1}{h^2}(b - 2a + b) = \frac{1}{h^2}(2b - 2a) = \frac{2}{h^2}(.0816 - .1566)R$$

$$= \frac{2R}{h^2}(- .0750) = - .1500 \frac{R}{h^2}$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2}(c - 2a + c) = \frac{1}{h^2}(2c - 2a) = \frac{2}{h^2}(.0725 - .1566)R$$

$$= \frac{2R}{h^2}(.0841) = - .1682 \frac{R}{h^2}$$

$$M_x \text{ pos LL} \approx - D\left(- .1500 \frac{R}{h^2} + (.15)(- .1682) \frac{R}{h^2}\right)$$

$$\approx -(- .1500 - .0252) \frac{D}{h^2} \frac{Wh^2}{D} = .1752W$$

$$\approx .1752(45000)$$

$$\approx 7884 \text{ ft-lb}$$

$$M_y \text{ pos LL} \approx - D\left(- .1682 \frac{R}{h^2} + (.15)(- .1500)\frac{R}{h^2}\right)$$

$$\approx - (- .1682 - .0225) \frac{D}{h^2} \frac{Wh^2}{D} = .1907W$$

$$\approx .1907(45000)$$

$$\approx 8582 \text{ ft-lb}$$

The Maximum Negative Moments occur at the center of each fixed panel edge, points 2,0 and 2,4. Although the Finite Difference Approximation analysis for the uniform loading condition indicated the grid spacing to be too coarse to closely approximate the maximum negative moments, the negative moments will be calculated for an order of magnitude determination and then checked with the Elastic Curve Plot.

$$M_{y\ neg} = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \cancelto{0}{\frac{\partial^2 w}{\partial x^2}}\right)$$

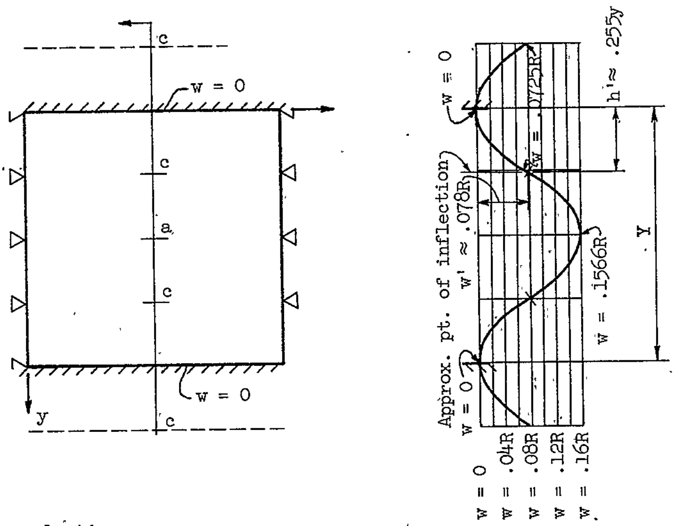
$$\frac{\partial^2 w}{\partial x^2} \approx \frac{1}{h^2}(c - 2(0) + c) = \frac{2c}{h^2} = \frac{2}{h^2}(.0725R)$$

$$= .1450\,\frac{R}{h^2}$$

$$M_{y\ neg\ LL} \approx -D\left(.1450\,\frac{R}{h^2}\right) = -\ .1450\,\frac{D}{h^2}\,\frac{Wh^2}{D}$$

$$\approx -\ .1450(45000)$$

Elastic Curve Plot for Negative Moment



From elastic

$$h' \approx .255Y$$

$$w' \approx .078R$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{(h')^2}(2w') = \frac{2(.078R)}{(.255)^2 Y^2} = \frac{2.40R}{Y^2}$$

with

$$h' = \frac{Y}{4},$$

$$R = \frac{Wh^2}{D} \doteq \frac{WY^2}{16D}$$

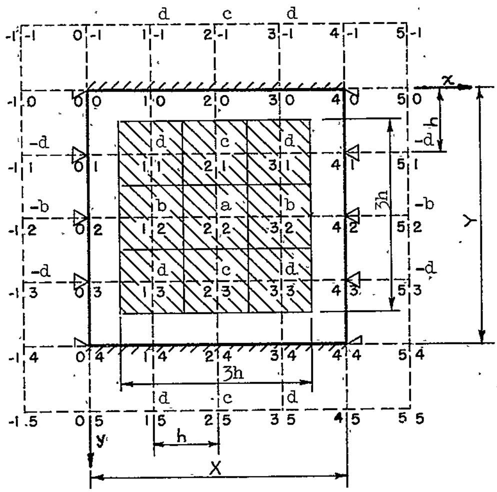
$$M_{y\ neg\ LL} \approx -D\left(2.40\,\frac{R}{Y^2}\right) = -\frac{2.40}{Y^2}\,D\,\frac{WY^2}{16D} = -\ .150W$$

$\approx -\ 6750$ ft-lb original calculation close enough

Finite Difference Approximation to Case "B" (see Appendix)

Concentrated Load Over All Internal Grid Points

$$h = \frac{X}{4} = \frac{Y}{4}$$



Boundary Conditions

Two opposite edges fixed
two opposite edges pinned

Therefore

$$\left.\begin{array}{l} w_{-1,y} = -w_{1,y} \\ w_{5,y} = -w_{3,y} \end{array}\right\} \text{due to allowing rotation of pinned edge}$$

$$\left.\begin{array}{l} w_{x,-1} = w_{x,1} \\ w_{x,5} = w_{x,3} \end{array}\right\} \text{due to rotation restraint}$$

$$w_{x,0} = w_{0,y} = w_{x,4} = w_{4,y} = 0$$

due to translation restraint

Since the deflection symmetry conditions are retained by the load placement, the same small alphabetic letters can be assigned to the points of common deflection.

The total live load previously placed on the slab will now be distributed over an area $3h \times 3h$, effectively concentrating a portion of the load over each internal grid point.

Previous total live load = 112.5 psf × 20' × 20' = 45000 lb = W

$$q_t = \frac{W}{(3h)^2} = \frac{45000}{9h^2} = \frac{5000}{h^2}$$

Superimposing the Finite Difference Approximation to the Biharmonic Equation over each point of differing deflection: Let $P = q_t \frac{h^4}{D} = 5000 \frac{h^2}{D}$

At grid point 2,2

$$20a - 8b - 8b - 8c - 8c + 2d + 2d + 2d + 2d = P$$

grouping common terms

$$20a - 16b - 16c + 8d = P \qquad (1)$$

At grid point 1,2; same for 3,2

$$20b - 8a - 8d - 8d + 2c + 2c + b - b = P$$

grouping common terms

$$-8a + 20b + 4c - 16d = P \qquad (2)$$

At grid point 2,3; same for 2,1

$$20c - 8a - 8d - 8d + 2b + 2b + c + c = P$$

grouping common terms

$$-8a + 4b + 22c - 16d = P \qquad)$$

At grid point 1,3; same for 1,1; 3,1 and 3,3

$$20d - 8b - 8c + 2a + d + d + d - d = P$$

grouping common terms

$$2a - 8b - 8c + 22d = P \qquad)$$

Placing the simultaneous equations in matrix form

$$\{A \,\vdots\, C\} = \left[\begin{array}{cccc:c} 2 & -8 & -8 & 22 & P \\ -8 & 20 & 4 & -16 & P \\ -8 & 4 & 22 & -16 & P \\ 20 & -16 & -16 & 8 & P \end{array}\right] \quad \begin{array}{c} \text{Sum} \\ 8+P \\ 0+P \\ 2+P \\ -4+P \end{array}$$

Note that the $\{A:C\}$ matrix above is identical with that of the original uniform loading consideration. Therefore the solutions to the simultaneous equations and the deflection coefficients are identical. The difference in the moments will come from the different load applied at the grid points.

Therefore

$a = .6312P$

$b = .4663P$

$c = .4146P$

$d = .3084P$

The equations for bending moments are (see F.D. Approx. for Case "A" uniform load)

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)$$

<u>The Maximum Positive Moments</u> occur at the center of the slab, point 2,2

$$\frac{\partial^2 w}{\partial x^2} \approx -.3298 \frac{P}{h^2} \qquad \frac{\partial^2 w}{\partial y^2} \approx -.4332 \frac{P}{h^2}$$

$$M_{x \text{ pos LL}} \approx -D\left(-.3298 \frac{P}{h^2} + (.15)(-.4332)\frac{P}{h^2}\right)$$

$$\approx .3948 \frac{D}{h^2} P = .3948 \frac{D}{h^2}\left(5000 \frac{h^2}{D}\right)$$

$$\approx 1974 \text{ ft-lb}$$

$$M_{y\ pos\ LL} \approx - D\left(- .4332\frac{P}{h^2} + (.15)(- .3298)\frac{P}{h^2}\right)$$

$$\approx .4827\frac{D}{h^2}P = .4827\frac{D}{h^2}\left(5000\frac{h^2}{D}\right)$$

$$\approx 2414 \text{ ft-lb}$$

The Maximum Negative Moments occur at the center of each fixed panel edge, points 2,0 and 2,4. Although the Finite Difference Approximation analysis for the uniform loading condition indicated the grid spacing to be too coarse to closely approximate the maximum negative moments, the negative moments will be calculated for an order of magnitude determination and then checked with the Elastic Curve Plot.

$$M_{y\ neg} = - D\left(\frac{\partial^2 w}{\partial y^2} + \mu \cancelto{0}{\frac{\partial^2 w}{\partial x^2}}\right)$$

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{1}{h^2}(c - 2(0) + c) = \frac{2c}{h^2} = \frac{2}{h^2}(.4146P) = .8292\frac{P}{h^2}$$

$$M_{y\ neg\ LL} \approx - D\left(.8292\frac{P}{h^2}\right) = - .8292\frac{D}{h^2}\left(5000\frac{h^2}{D}\right)$$

$$\approx - 4146 \text{ ft-lb}$$

The Elastic Curve Plot for this loading condition is the same as that previously shown for Case "B" uniform load. Therefore, the curve data at the inflection point is:

$$h' \approx .177Y$$

$$w' \approx .23P$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{17.88P}{Y^2}$$

with

$$h = \frac{Y}{4}, \quad P = \frac{5000h^2}{D} = \frac{5000Y^2}{16D}$$

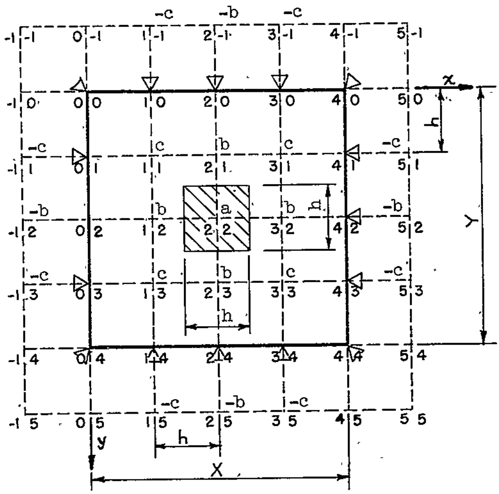
$$M_{y \text{ neg LL}} \approx -D\left(\frac{17.88P}{Y^2}\right) = -D\left(\frac{17.88}{Y^2}(5000)\ \frac{Y^2}{16D}\right)$$

$$\approx -5588 \text{ ft-lb}$$

Finite Difference Approximation to Case "C"

Concentrated Load Over Grid Point 2,2 (see Appendix)

$$h = \frac{X}{4} = \frac{Y}{4}$$



Boundary Conditions

All edges pinned

Therefore

$$\left.\begin{aligned} w_{-1,y} &= -w_{1,y} \\ w_{x,-1} &= -w_{x,1} \\ w_{5,y} &= -w_{3,y} \\ w_{x,5} &= -w_{x,3} \end{aligned}\right\} \text{due to allowing rotation of pinned edge}$$

$$w_{x,0} = w_{0,y} = w_{x,4} = w_{4,y} = 0$$

due to translation restraint

Since the deflection symmetry conditions are retained by the load placement, the same small alphabetic letters can be assigned to the points of common deflection.

The total live load previously placed on the slab will now be concentrated over an area $h \times h$ at point 2,2.

Previous total live load = 112.5 psf × 20' × 20' = 45000 lb = W

$$q_a = \frac{W}{h^2}$$

Superimposing the Finite Difference Approximation to the Biharmonic Equation over each point of differing deflection: Let $R = q_a \frac{h^4}{} = \frac{Wh^2}{D}$

At grid point 2,2

$$20a - 8b - 8b - 8b - 8b + 2c + 2c + 2c + 2c = R$$

grouping common terms

$$20a - 32b + 8c = R \tag{1}$$

At grid point 2,1; same for 1,2; 2,3 and 3,2

$$20b - 8a - 8c - 8c + 2b + 2b + b - b = 0$$

grouping common terms

$$-8a + 24b - 16C = 0 \tag{2}$$

At grid point 1,1; same for 1,3; 3,1 and 3,3

$$20c - 8b - 8b + 2a + c + c - c - c = 0$$

grouping common terms

$$2a - 16b + 20c = 0 \tag{3}$$

Placing the simultaneous equations in matrix form

$$\{A \vdots C\} = \begin{array}{c} \\ (3) \\ (2) \\ (1) \end{array} \begin{array}{c} \\ \left[\begin{array}{ccc:c} 2 & -16 & 20 & 0 \\ -8 & 24 & -16 & 0 \\ 20 & -32 & 8 & R \end{array}\right] \end{array} \begin{array}{c:c:} & \text{Sum} \\ \vdots & 6 \\ \vdots & 0 \\ \vdots & -4+R \end{array}$$

It should be noted that a similarity exists between the preceding simultaneous equations and those obtained under the uniform loading consideration. The only difference is in the value of the load placed on the grid points. Therefore, a large portion of $L\{T \vdots K\}$ matrix

$$L\{T \vdots K\} = \left[\begin{array}{ccc:c} 2 & -8 & 10 & 0 \\ -8 & -40 & -1.6000 & 0 \\ 20 & 120 & 12.80 & .0781R \end{array}\right] \begin{array}{c} \text{Check} \\ 3 \\ -.6000 \\ 1.0+.0781R \end{array}$$

Back substituting yields

$c = .0781R$

$b - 1.6000c = 0$

$b = 1.6000(.0781R)$

$b = .1250R$

$a - 8b + 10c = 0$

$a = 8(.1250R) - 10(.0781R) = 1.0000R - .7812R$

$a = .2188R$

Substituting back into original equations

(1) $20(.2188R) - 32(.1250R) + 8(.0781R)$ should $= 1.00R$

$4.3760R - 4.0000R + .6248R = 5.0008R - 4.0000R$

$= 1.0008R \sim 1.00R$

(2) $-8(.2188R) + 24(.1250R) - 16(.0781R)$ should $= 0$

$-1.7504R + 3.0000R - 1.2496R = 3.0000R - 3.0000R$

$= 0.0000R$

(3) $2(.2188R) - 16(.1250R) + 20(.0781R)$ should $= 0$

$.4376R - 2.0000R + 1.5620R = 1.9996R - 2.0000R$

$= - .0004R \sim 0$

Therefore the solutions satisfy the original equations fairly well.

The equations for bending moments are (see F.D. Approx. for Case "A" uniform load)

$$M_x = - D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = - D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)$$

The Maximum Positive Moments occur at the center of the slab, point 2,2. Therefore superimposing the Central Finite Difference Approximation to the second partial derivatives over point 2,2

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{1}{h^2}(b - 2a + b) = \frac{1}{h^2}(2b - 2a) = \frac{2}{h^2}(.1250 - .2188)R$$

$$= \frac{2R}{h^2}(- .0938) = - .1876 \frac{R}{h^2}$$

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2}(b - 2a + b) = - .1876 \frac{R}{h^2}$$

$$M_{y \text{ pos LL}} = M_{x \text{ pos LL}} \approx - D\left(- .1876 \frac{R}{h^2} + (.15)(- .1876)\frac{R}{h^2}\right)$$

$$\approx .1876 \frac{RD}{h^2}(1.15) = .2157 \frac{D}{h^2} R$$

$$\approx .2157 \frac{D}{h^2} \frac{Wh^2}{D} = .2157(45000)$$

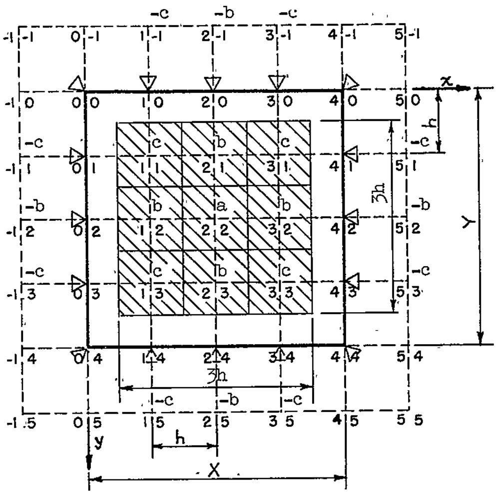
$$\approx 9706 \text{ ft-lb}$$

There is no negative moment involved in this case.

Finite Difference Approximation to Case "C" (see Appendix)

Concentrated Load Over All Internal Grid Points

$$h = \frac{X}{4} = \frac{Y}{4}$$



The total live load previously placed on the slab will now be distributed over an area $3h \times 3h$, effectively concentrating a portion of the load over each internal grid point.

Boundary Conditions

All edges pinned

Therefore

$$\left.\begin{aligned} w_{-1,y} &= -w_{1,y} \\ w_{x,-1} &= -w_{x,1} \\ w_{5,y} &= -w_{3,y} \\ w_{x,5} &= -w_{x,3} \end{aligned}\right\} \text{due to allowing rotation of pinned edge}$$

$$w_{x,0} = w_{0,y} = w_{x,4} = w_{4,y} = 0$$

due to translation restraint

Since the deflection symmetry conditions are retained by the load placement, the same small alphabetic letters can be assigned to the points of common deflection.

Previous total live load = 112.5 psf × 20' × 20' = 45000 lb = W

$$q_t = \frac{W}{(3h)^2} = \frac{45000}{9h^2} = \frac{5000}{h^2}$$

Superimposing the Finite Difference Approximation to the Biharmonic Equation over each point of differing deflection: Let $P = q_t \frac{h^4}{D} = 5000 \frac{h^2}{D}$

At grid point 2,2

$$20a - 8b - 8b - 8b - 8b + 2c + 2c + 2c + 2c = P$$

grouping common terms

$$20a - 32b + 8c = P \tag{1}$$

At grid point 2,1; same for 1,2; 2,3 and 3,2

$$20b - 8a - 8c - 8c + 2b + 2b + b - b = P$$

grouping common terms

$$-8a + 24b - 16c = P \tag{2}$$

At grid point 1,1; same for 1,3; 3,1 and 3,3

$$20c - 8b - 8b + 2a + c + c - c - c = P$$

grouping common terms

$$2a - 16b + 20c = P \tag{3}$$

Placing the simultaneous equations in matrix form

$$\{A \vdots C\} = \begin{array}{c} (3) \\ (2) \\ (1) \end{array} \left[\begin{array}{ccc:c} 2 & -16 & 20 & P \\ -8 & 24 & -16 & P \\ 20 & -32 & 8 & P \end{array}\right] \vdots \begin{array}{c} \text{Sum} \\ 6+P \\ 0+P \\ -4+P \end{array} \vdots$$

Note that the $\{A \vdots C\}$ matrix above is identical with that of the original uniform loading consideration. Therefore the solutions to the simultaneous equations and the deflection coefficients are identical. The difference in the moments will come from the different loads applied at the grid points.

Therefore

$a = 1.0310P$

$b = .7500P$

$c = .5469P$

The equations for bending moments are (see F.D. Approx. for Case "A" uniform load)

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right)$$

<u>The Maximum Positive Moments</u> occur at the center of the slab, point 2,2.

$$\frac{\partial^2 w}{\partial x^2} \approx -.5620 \frac{P}{h^2} \qquad \frac{\partial^2 w}{\partial y^2} \approx -.5620 \frac{P}{h^2}$$

$$M_{y \text{ pos LL}} = M_{x \text{ pos LL}} \approx -D\left(-.5620 \frac{P}{h^2} + (.15)(-.5620) \frac{P}{h^2}\right)$$

$$\approx .5620\ D \frac{P}{h^2} (1.15)$$

$$\approx .6463 \frac{D}{h^2} (5000) \frac{h^2}{D}$$

$$\approx 3232 \text{ ft-lb}$$

There is no negative moment involved in this case.

VIII. SUMMARY OF RESULTS OBTAINED BY THE FINITE DIFFERENCE APPROXIMATION

Case "A"

A = area loaded	$(4h)^2$	$(3h)^2$	$(h)^2$
A/A total	1.0000	.5625	.0625
Positive Moment point 2,2			
$M_{x\&y}$ pos DL	+762	+762	+762
$M_{x\&y}$ pos LL	+979	+1741	+7565
$M_{x\&y}$ pos	+1741	+2503	+8327
Negative Moment (center of fixed panel edge)			
$M_{x\&y}$ neg DL	-1662*	-1662*	-1662*
$M_{x\&y}$ neg LL	-2138*	-3803*	-5139
$M_{x\&y}$ neg	-3800*	-5465	-6801

Case "B"

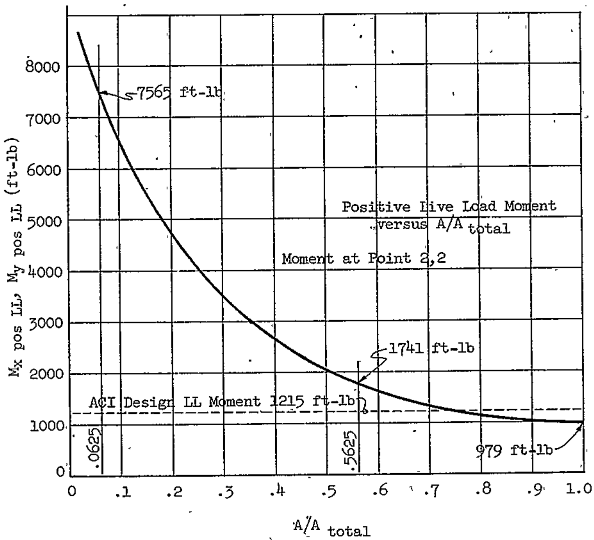
A = area loaded	$(4h)^2$	$(3h)^2$	$(h)^2$
A/A total	1.0000	.5625	.0625
Positive Moment point 2,2			
M_x pos DL	+864	+864	+864
M_x pos LL	+1110	+1974	+7884
M_x pos	+1974	+2818	+8748
M_y pos DL	+1056	+1056	+1056
M_y pos LL	+1358	+2414	+8582
M_y pos	+2414	+3470	+9638
Negative Moment (center of fixed panel edge)			
M_y neg DL	-2444*	-2444*	-2444*
M_y neg LL	-3143*	-5588*	-6525
M_y neg	-5587*	-8032	-8969

Case "C"

A = area loaded	$(4h)^2$	$(3h)^2$	$(h)^2$
A/A total	1.0000	.5625	.0625
Positive Moment point 2,2			
$M_{x\&y}$ pos DL	+1414	+1414	+1414
$M_{x\&y}$ pos LL	+1818	+3232	+9706
$M_{x\&y}$ pos	+3232	+4646	+11120

An * indicates that the results were obtained using the Elastic Curve Plot

Table 2. Maximum Moments Obtained by the Finite Difference Approximation for the Three Conditions of Loading



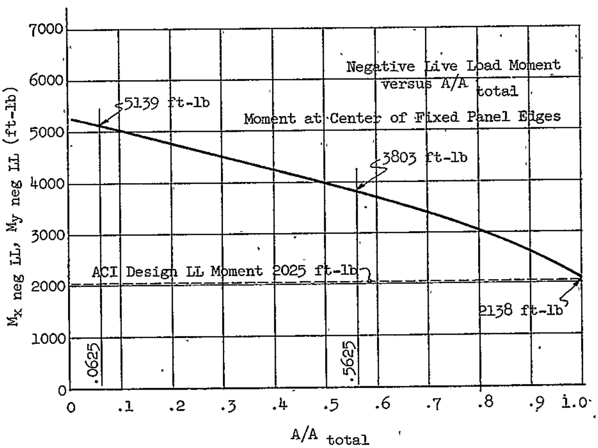
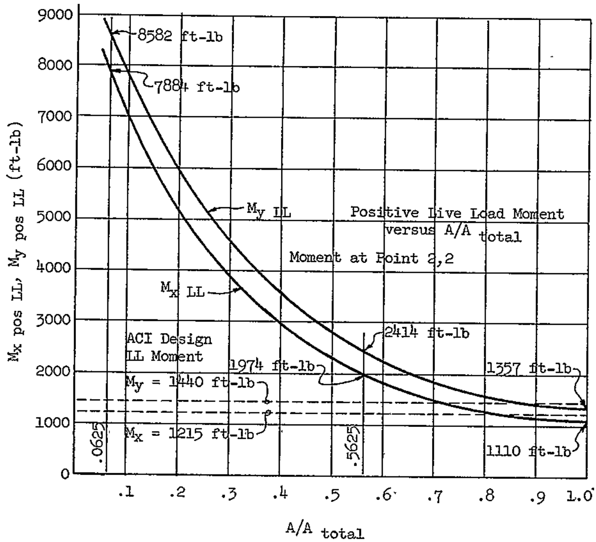


Figure 2.- Case "A" Live Load Moments Versus Ratio of Load Area to Total Area.



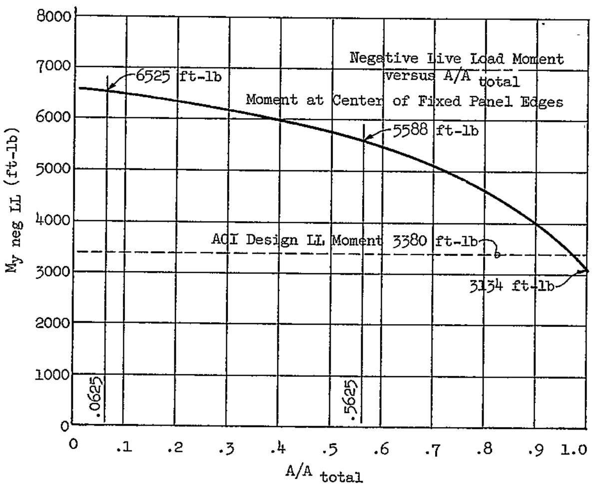


Figure 3.- Case "B" Live Load Moments Versus Ratio of Loaded Area to Total Area.

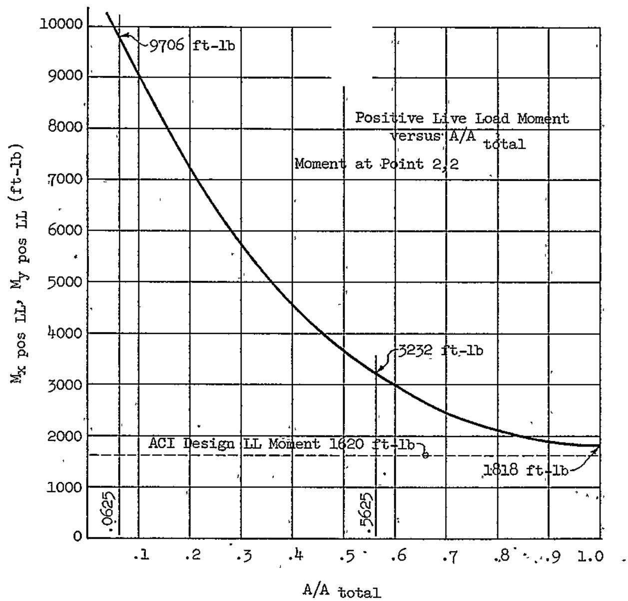


Figure 4.- Case "C" Live Load Moments Versus Ratio of Loaded Area to Total Area.

A study of the plots of the Live Load Moments versus A/A_{total} leads to an interesting possibility for approximating a maximum concentrated live load that could be placed on the slab without exceeding the ACI design moment.

In the cases examined the design live load is known to be 112.5 psf. Therefore the total live load for this particular slab is 112.5 psf $\times$ 20' $\times$ 20' = 45000 lb. Suppose we are considering placing a load at the center of a Case "A" type slab, to bear over an area 5 ft $\times$ 5 ft. A/A_{total} = (5 ft $\times$ 5 ft)/(20 ft $\times$ 20 ft) = .0625. In Case "A", the ACI design live load moment is 1215 ft-lb. The plot of Case "A" shows that a 45000 lb live load, at A/A total = .0625, yields a moment of 7565 ft-lb.

Therefore the approximate allowable concentrated load

$$(T) = \frac{1215}{7565}(45000 \text{ lb})$$

$$= 7220 \text{ lb}$$

Since the equation for the moment has been determined in the preceding analysis for the load over $h \times h$ and A/A_{total} = .0625, the new moment can be checked.

Case "A" load over $h \times h$ $M_{pos} = .1681T$

$$= .1681(7220) = 1215 \text{ ft-lb}$$

Therefore the ACI design moment has not been exceeded but it should be noted that this concentrated load does not allow for the placement of any additional live load on the slab.

After the maximum concentrated load for the positive moment has been established, it is necessary to check the maximum allowable load related to the negative moment.

In a manner similar to above, $T = \frac{2025}{5139}$ (45000 lb) = 17720 lb

However, the lesser of the two allowable loads calculated would be the maximum load that could be placed at the slab center without exceeding the ACI design moment.

It is important to recall that the load placed on the slab, for the analysis, produced a symmetry in the deflections. This symmetry must be maintained to use the curves in determining the maximum allowable concentrated load. It would not be maintained if, for example, the total load was placed one grid point away from the slab center. In a case of that nature it would be necessary to use the Finite Difference Approximation to analyze the slab with the load in that particular position.

The curve of the Live Load Moments versus A/A_{total} indicates that the maximum concentrated load could be determined for all values of A/A_{total}. For this study only three points were used to determine the curve. Thus, it should be realized that the curve between these points is only roughly approximated. To obtain additional points for this curve, it would be necessary to reduce the grid size. A reduction in the grid size would increase the number of simultaneous equations necessary to solve but it would also increase the accuracy of the approximation.

IX. CONCLUSION

Although the large grid network selected for the analysis tends to lessen the accuracy of the approximation, it also effectively reduces the number of simultaneous equations which are necessary to solve. A further reduction in the number of simultaneous equations was obtained from symmetry, by identifying grid points of common deflection. Table 1 shows that, for a uniformly loaded slab, the Finite Difference Approximation to the Biharmonic Equation and ACI 318-63 Method 3 yield comparable results for the positive moment. In the area of the fixed panel edges, the Finite Difference Approximation did not yield comparable answers. This was because a grid point did not fall near the inflection point of the elastic curve. It was determined that the location of the inflection point could be reasonably approximated by plotting the calculated deflections and sketching the elastic curve through these points. The negative moment was then recalculated using the data from this Elastic Curve Plot. Therefore, the combination of the large grid Finite Difference Approximation to the Biharmonic Equation and the Elastic Curve Plot produced an efficient and reasonably close approximation to the ACI requirements.

Since a reasonable comparison was obtained between the two methods, the Finite Difference Approximation to the Biharmonic Equation was used to determine the design moments for the slab under the influence of concentrated loads. Table 2 and Figures 2, 3, and 4 show how the maximum moments vary as the total live load is expanded from application

over an area $\frac{X}{4} \times \frac{Y}{4}$ to a full uniform load. In Chapter VIII, a method is shown for approximating the maximum allowable concentrated live load which can be applied to the slab without exceeding the ACI design moments. It should be noted that for this study the curves in figures 2, 3, and 4 were drawn through only three points. Therefore, any value picked from the curves, other than those calculated, must be considered only as a very rough approximation. It should also be noted that although the maximum allowable concentrated load does not produce calculated moments which exceed the ACI design moments, the results have not been proven conclusively by testing.

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FINITE DIFFERENCE APPROXIMATION TO THE BIHARMONIC EQUATION

The equation of equilibrium for a homogeneous, isotropic flat plate in terms of its deflection is

$$D \nabla^4 w = q(x,y) \qquad \boxed{1}$$

where

$q(x,y)$ = loading function

$D = \dfrac{Et^3}{12(1 - \mu^2)}$

E = Young's modulus of elasticity

t = thickness of plate

μ = Poisson's ratio

$\nabla^4 w$ = biharmonic equation

The biharmonic operator $\nabla^4 = \dfrac{\partial^4}{\partial x^4} + \dfrac{2\partial^4}{\partial x^2 \partial y^2} + \dfrac{\partial^4}{\partial y^4}$

$$\nabla^4 = \nabla^2(\nabla^2) = \nabla^2 \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{\substack{\text{Laplacian} \\ \text{Operator}}}$$

The biharmonic operator operates on the deflection (w) as does the Laplacian operator. Thus,

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

Now, $\frac{\partial^2 w}{\partial x^2}$ is represented by the Central Finite Difference Approximation as

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{1}{h^2} \left(w_{i-1,j} \quad - 2w_{i,j} \quad w_{i+1,j} \right)$$

and

$$\frac{\partial^2 w}{\partial y^2} \approx \frac{1}{h^2} \begin{Bmatrix} w_{i,j-1} \\ -2w_{i,j} \\ w_{i,j+1} \end{Bmatrix}$$

Therefore, $\nabla^2 w$ is represented by

$$\nabla^2 w \approx \left[\frac{1}{h^2} \begin{Bmatrix} & w_{i,j-1} & \\ w_{i-1,j} & -4w_{i,j} & w_{i+1,j} \\ & w_{i,j+1} & \end{Bmatrix} \right], \qquad [2]$$

In modular form

$$\nabla^2 w \approx \frac{1}{h^2} \begin{Bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{Bmatrix}$$

Now let $\phi = \nabla^2 w$. Thus, $\nabla^4 w = \nabla^2(\nabla^2 w) = \nabla^2 \phi$, $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$.

Since this is the same differential equation as for the deflection, it can be concluded that

$$\nabla^2\phi \approx \frac{1}{h^2}\left\{ \begin{array}{ccc} & \phi_{i,j-1} & \\ \phi_{i-1,j} & -4\,\phi_{i,j} & \phi_{i+1,j} \\ & \phi_{i,j+1} & \end{array} \right\} \qquad [3]$$

Each of the encircled modules in [3] represents the entire module within the [ ] brackets of [2]. Therefore, [2] must be superimposed on each module of [3], which follows:

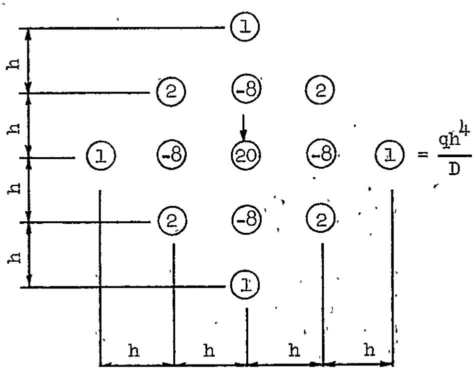
$$\nabla^4 w \approx \nabla^2\phi \approx \frac{1}{h^2}\left\{\frac{1}{h^2}\right\{$$

i-2	i-1	i	i+1	i+2	
		(1)(1) = 1			j-2
	(1)(1) + (1)(1) = 2	(-4)(1) + (1)(-4) = -8	(1)(1) + (1)(1) = 2		j-1
(1)(1) = 1	(-4)(1) + (1)(-4) = -8	(-4)(-4) + (1)(1) + (1)(1) + (1)(1) + (1)(1) = 20	(-4)(1) + (1)(-4) = -8	(1)(1) = 1	j
	(1)(1) + (1)(1) = 2	(-4)(1) + (1)(-4) = -8	(1)(1) + (1)(1) = 2		j+1
		(1)(1) = 1			j+2

Therefore, the Central Finite Difference Approximation to the Biharmonic Equation is represented by the module

$$\nabla^4 w \approx \frac{1}{h^4}\left\{\begin{array}{ccccc} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{array}\right\}$$

With this Finite Difference Approximation module centrally superimposed over each point of different deflection on the plate grid system, the simultaneous equations are developed in the form



Cholesky or Crout Method for Solving Simultaneous Equations[13]

Given a set of simultaneous equations in the form

$$AX - C = 0,$$

we want to get this set of simultaneous equations in a solved fashion such that $TX - K = 0$ where T = upper unit triangular matrix

$$T = \begin{bmatrix} 1 & t_{12} & t_{13} & t_{14} & \cdots & & t_{1n} \\ 0 & 1 & t_{23} & t_{24} & \cdots & & t_{2n} \\ 0 & 0 & 1 & t_{34} & \cdots & & t_{3n} \\ \cdot & \cdot & \cdot & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & t_{(n-1)n} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

and K is a column vector

$$K = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \cdot \\ \cdot \\ \cdot \\ k_n \end{bmatrix}$$

To obtain this solution for the given simultaneous equations, multiply $TX - K = 0$ by a lower triangular matrix L such that

$$L = \begin{bmatrix} l_{11} & 0 & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & l_{44} & 0 \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ l_{n1} & l_{n2} & l_{n3} & l_{n4} & & l_{nn} \end{bmatrix}$$

A necessary condition for this operation is that both T and L must be non-singular. Therefore,

$$L\left\{TX - K\right\} = 0$$

or

$$LTX - LK = 0$$

Comparing this to original problem, we see that

$$LT = A$$

$$LK = C$$

Partitioning these matrices to aid in their direct solution

$$\left\{A \vdots C\right\} = L\left\{T \vdots K\right\}.$$

Expanding these matrices to their coefficients

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\cdots a_{1n} & \vdots & c_1 \\ a_{21} & a_{22} & a_{23}\cdots a_{2n} & \vdots & c_2 \\ a_{31} & a_{32} & a_{33}\cdots a_{3n} & \vdots & c_3 \\ \cdot & \cdot & \cdot & \vdots & \cdot \\ \cdot & \cdot & \cdot & \vdots & \cdot \\ \cdot & \cdot & \cdot & \vdots & \cdot \\ a_{n1} & a_{n2} & a_{n3}\cdots a_{nn} & \vdots & c_n \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & \cdots 0 \\ l_{21} & l_{22} & 0 & \cdots 0 \\ l_{31} & l_{32} & l_{33} & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ l_{n1} & l_{n2} & l_{n3} & \cdots l_{nn} \end{bmatrix} \begin{bmatrix} 1 & t_{12} & t_{13}\cdots & & t_{1n} & \vdots & k_1 \\ 0 & 1 & t_{23}\cdots & & t_{2n} & \vdots & k_2 \\ 0 & 0 & 1 & \cdots & t_{3n} & \vdots & k_3 \\ \cdot & \cdot & \cdot & & \cdot & \vdots & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \vdots & \cdot \\ \cdot & \cdot & 0 & 1 & t_{(n-1)n} & \vdots & \cdot \\ 0 & 0 & 0 & 0 & 1 & \vdots & k_n \end{bmatrix}$$

The solution of the coefficients of the L and T matrices follows a systematic procedure as shown in equation form:

$$l_{i1} = a_{i1} \qquad\qquad t_{1j} = \frac{a_{1j}}{l_{11}}$$

$$l_{ij} = a_{ij} - \sum_{r=1}^{r=j-1} l_{ir}t_{rj} \qquad\qquad t_{ij} = \frac{1}{l_{ii}}\left[a_{ij} - \sum_{r=1}^{r=i-1} l_{ir}t_{rj}\right]$$

One of the great advantages to using this method for any hand operation is that it lends itself to a Check column.

By summing up all coefficients in each row of the $\{A\vdots C\}$ matrix, placing each sum adjacent to the row it represents, then a Check column equal to all the coefficients in the $\{T\vdots K\}$ matrix is obtained by the equation

$$\text{Check}_i = \frac{1}{l_{ii}}\left[\text{Sum}_i - \sum_{r=1}^{r=i-1} l_{ir}\ \text{Check}_r\right]$$

Therefore, as a final matrix form we have

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{1n} & \vdots\, c_1 \\ a_{21} & a_{22} & a_{23} & \cdots a_{2n} & \vdots\, c_2 \\ & & & & \vdots \\ & & & & \vdots \\ & & & & \vdots \\ a_{n1} & & & \cdot\ a_{nn} & \vdots\, c_n \end{bmatrix}
\begin{array}{c} \text{Sum} \\ \begin{bmatrix} \text{Sum}_1 \\ \text{Sum}_2 \\ \cdot \\ \cdot \\ \cdot \\ \text{Sum}_n \end{bmatrix} \end{array}
=
\begin{bmatrix} l_{11} & t_{12} & t_{13} & \cdots t_{1n} & \vdots\, k_1 \\ l_{21} & l_{22} & t_{23} & \cdot & \vdots\, k_2 \\ l_{31} & l_{32} & l_{33} & \cdot & \vdots\, k_3 \\ \cdot & & & & \vdots \\ \cdot & & & & \vdots \\ l_{r} & & & l_{nn} & \vdots\, k_n \end{bmatrix}
\begin{array}{c} \text{Check} \\ \begin{bmatrix} \text{Check}_1 \\ \text{Check}_2 \\ \text{Check}_3 \\ \cdot \\ \cdot \\ \text{Check}_n \end{bmatrix} \end{array}
$$

Note that with this method of combining the L and T matrix the 1.0's in the diagonal of the T matrix are not written in. Therefore, when comparing the results of the Check column, +1.0 must be added to the coefficients of the $\{T \vdots K\}$ matrix to obtain the proper check.

By back substituting, the solution to the simultaneous equations are obtained. That is,

$$X_n = k_n$$

$$X_{(n-1)} + t_{(n-1)n}X_n = k_{(n-1)}$$

$$X_{(n-2)} + t_{(n-2)(n-1)}X_{(n-1)} + t_{(n-2)n}X_n = k_{(n-2)}$$

etc.

As a final check, after the complete solution has been obtained, the values for X_i should be substituted back into all of the original equations